Enhanced variational approach for damage analysis of laminated composite

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Abstract

The main objective of this research is to introduce a new coupled analytical-numerical procedure to calculate stiffness reduction due to matrix cracks in composite laminates subjected to general inplane loading conditions. A novel micro-model is developed for transverse cracking of general symmetric laminates based on a variational approach. The proposed method overcomes the restrictions of previous similar approaches regarding damage in laminates with general symmetric stacking sequences. A polynomial series is used for calculating the stress perturbation due to intralaminar cracks in conjunction with the principle of minimum complementary energy. The numerical part deals with well-known Newton-Raphson procedure to solve the governing equations for the unknown parameters for the stress functions. Furthermore, the analytical part is can be solved just one time by a symbolic calculation for unknown geometry parameters, material properties and loading conditions, and thus be attractive for implementation in finite element software. Comparison between existing experimental and analytical results and the results of the present study are are found to be in good agreement.

1. Introduction

Transverse or matrix cracking is usually the first type of damage which occurs in composite laminates with off-axis plies. Although transverse cracking is not itself hugely problematic, induced delaminations, leakage in pressure vessels, reduction of the fatigue life, and providing paths for chemical gas or liquids that can deteriorate the reinforcing fibers are side effects that provide justification to conduct more research involving such type of damage.

The simplest approach for evaluating reduced stiffness of damaged laminates is the ply discount method [1-3] but it may overestimate the reduced stiffness because it does not account for the residual stiffness of the damaged laminas, which is not negligible even when crack saturation has been reached [4, 5]. More accurate prediction of stiffness reduction is possible with analytical models accounting for the stress transfer among laminas. These models include shear-lag methods [6-8], meso-scale methods [9-12], finite element method (FEM) [13], continuum damage mechanics [14] , discrete damage mechanics [15-21], variational approaches [22-29] and stress transfer [30, 31]. Further, crack separation models [32, 33] and synergistic damage mechanics models [34, 35] combines characteristics of the above and two-scale numerical models [36-40] are more computational expensive but include more details and less restricting assumptions. While most analytical models assume equally spaced cracks, non-uniform distributions of transverse matrix cracks have been studied [41-47]. Combining the Continuum Damage Mechanics (CDM) with Discrete Damage Mechanics (DDM) leads to prediction of progressive damage in composite laminates. In combined CDM/DDM models, damage parameters of a homogenized model are calculated from micro modeling, i.e., a discrete damage model. Barbero et al. combined the shear-lag method with CDM for progressive damage analyses of general symmetric laminates [15, 16, 36, 37, 39, 48, 49] and general unsymmetric laminates [50]. Ghayour et al. developed a multi-scale approach in which both micro and macro models were based on the FE approach [40]. A combination of the stress transfer method with CDM and the layer-wise method was utilized for transverse crack analysis of general symmetric laminates [51] and thermo-elastic analysis of cracked laminates [52].

The variational approach was first used by Hashin for stiffness reduction analysis of cracked cross-ply laminates under tensile and shear loadings [22] and by Nairn and colleagues for thermo-elastic analysis of transverse cracking and also induced delamination [23-27]. The main concept of this variational approach is to calculate the stress distribution around a crack by minimizing the complementary energy of the laminate in presence of the matrix cracks. During the last two decades, the results from variational approach have shown good comparison with experimental observations. Many researchers have attempted to develop the variational approach for cross-ply [53-57] and general angle-ply laminates under general loading conditions [54-56].

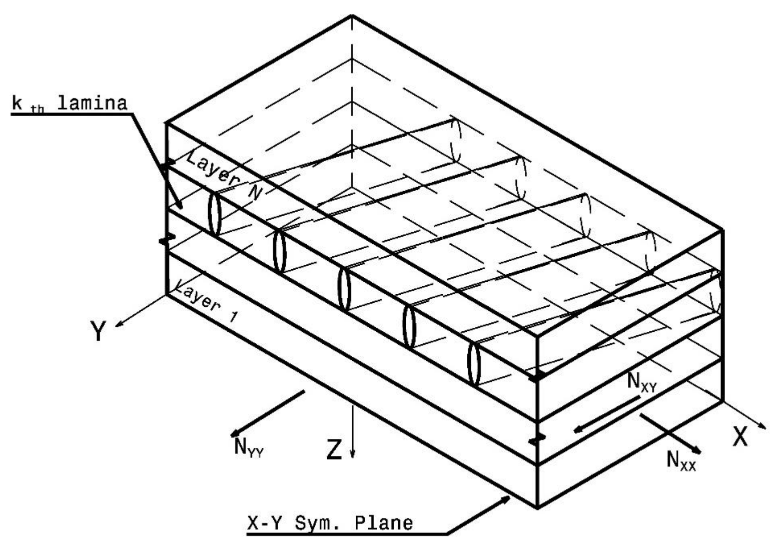
The proposed method is an enhancement of [9] because it is able to calculate stiffness reduction due to matrix cracking of general symmetric laminates.

The main objective of this work is to introduce a general approach for damage analysis of composite laminates. The main advantage of this approach is removing the restrictions of theoretical approaches, which are usually applicable only to specific conditions, but instead utilizing the generality of numerical procedures. In other words, this approach is a coupled analytical-numerical procedure including the benefits and excluding the weaknesses of both analytical and numerical procedures. For this purpose, a variational approach is developed for calculating stiffness reduction of general symmetric laminates. The novelty of this work is utilizing polynomial series for the stress perturbation in order to satisfy boundary conditions and equilibrium equations for general symmetric laminates. Unknown degrees of freedoms of polynomial series are calculated with the principle of minimum complementary energy. The resulting equations are solved using the

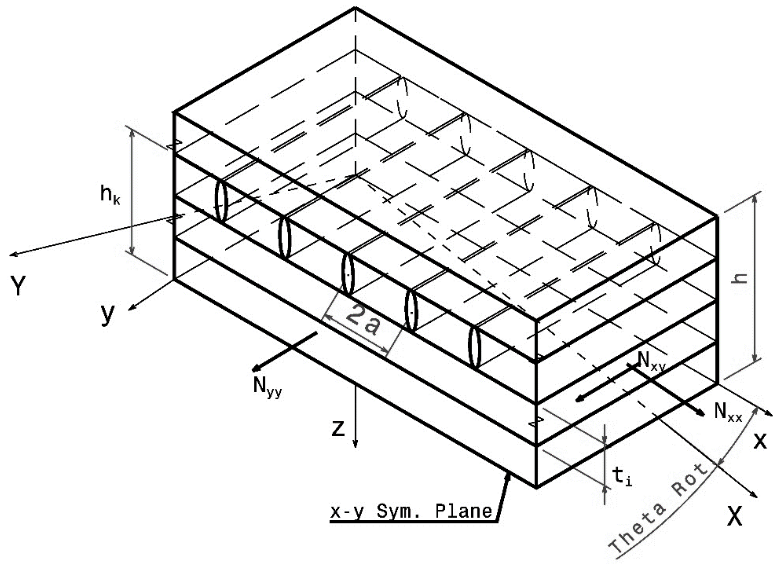
Newton–Raphson iterative technique. Polynomial series are advantageous due to the simplicity of using them in mathematical operations such as differentiation and integration that are necessary for minimizing the complementary energy. Furthermore, the proposed method can be formulated and incorporated for commercial FEM software. An example is presented in Section 3.2. Although the main concept of the variational approach using polynomial series has been previously applied for stability analysis of composite laminates [58-60], it is here developed for the first time for transverse cracking analysis of laminates.

2. Formulation

Considering a general symmetric laminate with a total of 2N layers, numbered as illustrated in Fig. 1, which depicts only the top portion of the laminate. The laminate contains a matrix crack at k-th lamina (mirrored in its symmetric counterpart lamina). The laminate is then subjected to external membrane loads N_{XX}, N_{YY}, and N_{XY} acting on the whole laminate in the laminate coordinate system (XYZ) as shown in Fig. 1.(a). A rotation ($\theta^{(k)}$-90) around the Z-axis results in a laminate $[(\theta^{(1)} - \theta^{(k)} + 90)_m, \ldots, (90)_n, (\theta^{(k+1)} - \theta^{(k)} + 90)_p, \ldots, (\theta^{(n)} - \theta^{(k)} + 90)_r]_s$ laminate. Thus, a series of parallel cracks are formed in the k-th lamina, which after rotation becomes a (90°) lamina in cracked laminate coordinate system (xyz) (Fig. 1.(b)).



(a)



(b)

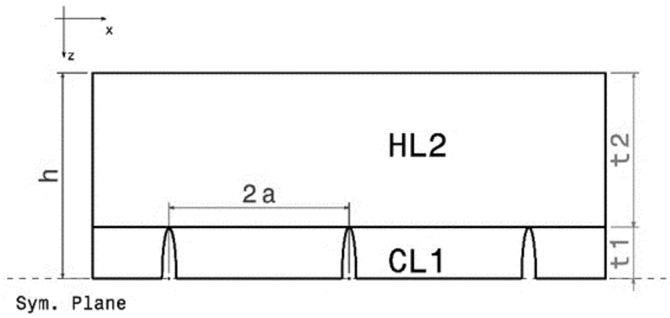
Fig. 1. Laminate with internal matrix crack (a) Original coordinate system (b) Cracked coordinate system.

Next, the stress can be represented as the sum of un-cracked stress plus a perturbation due to the presence of the cracks, as follows:

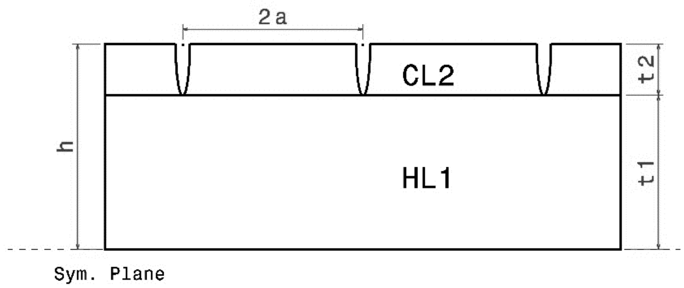
$$\sigma_{kl}^{(i)}(x) = \sigma_{kl}^{0(i)}(x) + \sigma_{kl}^{(i)}(x) \tag{1}$$

where, superscript 0 denotes the stress in un-cracked laminate, *k* and *l* stand for x, y, z in cracked coordinate system, and the last term is the stress perturbation, which is a function of *x*.

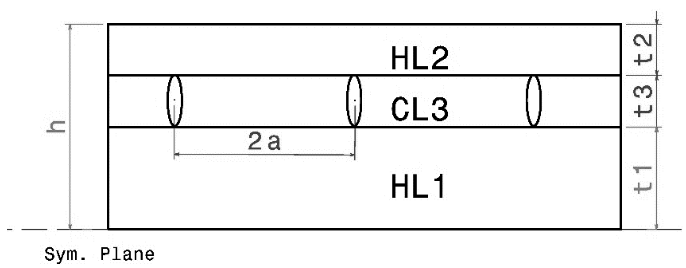
According to the location of cracked layer, it can be reperesentative of mid-ply crack, and internal or an external crack. Fig. 2.(a) shows a general symmetric laminate containing matrix crack at the middle-ply (CL1). For this case, the rest of laminas are assumed to be a homogenized layer with the equivalent stiffness (HL2). Thus, equilibrium functions and boundary conditions can be applied with N_{eq} =2 which means two series of stress functions are required to estimate the stress distribution of cracked laminate. The same assumption has been considered for the laminate with crack at top layer (CL2) with middle equivalent homogenized layer (HL1) as shown in Fig. 2.(b). Furthermore, in order to calculate the stress perturbation for internal cracked lamina, two homogenized equivalent laminas are required at top and bottom of the cracked layer (HL1 and HL2) (Fig. 2.(c)). So, for this case, three equivalent layers N_{eq} =3 are required to find stress perturbation due to matrix cracking.



(a)



(b)



(c)

Fig. 2 Symmetric laminate containing matrix cracks at (a) Middle-ply (CL1) (b) Top layer (CL2) (c) Internal layer (CL3)

For a symmetric laminate with total $2N_{eq}$ layers, stress perturbation functions define as:

$$\sigma_{xx}^{(i)} = -\phi_i(x)$$
$$\sigma_{yy}^{(i)} = -\psi_i(x) \qquad (2)$$

$\sigma_{xy}^{(i)} = -\eta_i(x)$by inserting Eq.2 into equilibrium condition ($\sigma_{ij,j} = 0$), following equations are derived:

$$-\phi_i'(x) + \sigma_{xz,z}^{(i)} = 0$$

$$- \psi_i(x) + \sigma_{yz,z}^{(i)} = 0$$
$$\sigma_{xz,x}^{(i)} + \sigma_{zz,z}^{(i)} = 0 \quad (3)$$

So, the solution for shear stresses could be found as:

$$\sigma_{xz}^{(i)} = -\phi_i'(x)\, z + F_i(x)$$
$$\sigma_{yz}^{(i)} = -\psi_i(x)\, z + H_i(x) \quad (4)$$
$$\sigma_{zz}^{(i)} = -(\phi_i''(x)\frac{z^2}{2} + F_i'(x)Z + G_i(x))$$

in which, F(x), G(x) and H(x) are unknown functions of *x* that needs to be found with consideration of boundary conditions of the cracked laminate.

Boundary conditions for the laminates containing matrix crack at 'k'th layer, is as follow:

$$\sigma_{xx}^{(k)}(\pm a, z) = -\sigma_{xx}^{0\,(k)}$$
$$\sigma_{xy}^{(k)}(\pm a, z) = -\sigma_{xy}^{0\,(k)} \quad (5)$$
$$\sigma_{xx}^{(k)}(\pm a, z) = 0$$

where, *a* is half crack length. Zero traction at top surface may lead to following functions:

$$\sigma_{xz}^{(N_{eq})}(x, z = h) = 0$$
$$\sigma_{yz}^{(N_{eq})}(x, z = h) = 0 \quad (6)$$
$$\sigma_{zz}^{(N_{eq})}(x, z = h) = 0$$

Furthermore, because of symmetric condition, following stresses should be zero.

$$\sigma_{xz}^{(1)}(x, z = 0) = 0 \quad (7)$$
$$\sigma_{yz}^{(1)}(x, z = 0) = 0$$

Considering traction continuity, one can find:

$$\sigma_{xz}^{(i)}(x, z = h_i) = \sigma_{xz}^{(i+1)}(x, z = h_i) \quad \text{for i} = 1{:}\,\mathrm{N_{eq}} - 1$$
$$\sigma_{yz}^{(i)}(x, z = h_i) = \sigma_{yz}^{(i+1)}(x, z = h_i) \quad \text{for i} = 1{:}\,\mathrm{N_{eq}} - 1 \quad (8)$$
$$\sigma_{zz}^{(i)}(x, z = h_i) = \sigma_{yz}^{(i+1)}(x, z = h_i) \quad \text{for i} = 1{:}\,\mathrm{N_{eq}} - 1$$

Assuming that the laminate is infinite in *y*-direction (i.e., away from free edges). For uncracked laminate, force equilibrium at *x* direction leads to:

$$N_{xx} = \int_{-h}^{h} \sigma_{xx} dz = 2\sum_{i=1}^{N_{eq}} \sigma_{xx}^{0(i)} \mathrm{t_i} \quad (9)$$

where t_i is the thickness of the 'i'th lamina. While for a cracked laminate, the following equation is valid:

$$N_{xx} = \int_{-h}^{h} \sigma_{xx} dz = 2\sum_{i=1}^{N_{eq}} \sigma_{xx}^{0(i)} \mathrm{t_i} + 2\sum_{i=1}^{N_{eq}} \sigma_{xx}^{(i)} \quad (10)$$

So, considering Eq.9 and Eq.10, one can find:

$$\sum_{i=1}^{N_{eq}} \sigma_{xx}^{(i)} = -\sum_{i=1}^{N_{eq}} \phi_i(x)\mathrm{t_i} = 0 \quad (11)$$

The same result for N_{xy} and N_{yy} can be found as:

$$\sum_{i=1}^{N_{eq}} \psi_i(x)\mathrm{t_i} = 0 \quad (12)$$

$$\sum_{i=1}^{N_{eq}} \eta_i(x)\mathrm{t_i} = 0$$

Based on Eq.11 and 12, it could be understood that for laminate with $2N_{eq}$ layers, $3*(N_{eq}-1)$ stress functions are required to predict the stress function of cracked laminates. Further sections deal with finding the stress functions for mid-plane, external, and internal cracked layers. Thus, having the stress functions for these three cases, completes the derivation of all the stress functions needed for general symmetric laminates.

2.1. Mid-plane cracks

For a general symmetric laminate containing center crack (k = $1N_{eq}$ = 2) as shown in Fig. 2.(a), Eq.5 is defined as:

$$\sigma_{xx}^{(1)}(\pm a, z) = -\sigma_{xx}^{0\,(1)}$$
$$\sigma_{xy}^{(1)}(\pm a, z) = -\sigma_{xy}^{0\,(1)} \quad (13)$$
$$\sigma_{xz}^{(1)}(\pm a, z) = 0$$

In addition, Eq.11 and Eq.12 can be summarized as:

$$\phi_2(x) = -\frac{1}{\lambda_1}\phi_1(x)$$
$$\psi_2(x) = -\frac{1}{\lambda_1}\psi_1(x) \quad (14)$$

$$\eta_2(x) = -\frac{1}{\lambda_1}\ \eta_1(x)$$
$$\lambda_1 = \frac{t_2}{t_1}$$

So, following Eq.6 to 8 as well as Eq.13 and 14, one can find the stress tensor for the (1) “center”CL1 and (2) homogenized outer laminas (HL2) as follows (for more details, see [55]):

$$\sigma_{xx}^{(1)}(x) = -\phi(x); \qquad \sigma_{xy}^{(1)}(x) = -\psi(x); \qquad \sigma_{yy}^{(1)}(x) = -\eta(x)$$
$$\sigma_{xz}^{(1)}(x) = \phi'(x)\,z; \qquad \sigma_{yz}^{(1)}(x) = \psi'(x)\,z; \qquad \sigma_{zz}^{(1)}\,(z) = \frac{(ht_1 - z^2)}{2}\phi''(x) \tag{15}$$

$$\sigma_{xx}^{(2)}(x) = \frac{1}{\lambda_1}\phi(x); \qquad \sigma_{xy}^{(2)} = \frac{1}{\lambda_1}\psi(x); \qquad \sigma_{yy}^{(2)}(x) = \frac{1}{\lambda_1}\eta(x) \tag{16}$$
$$\sigma_{xz}^{(2)} = \frac{1}{\lambda_1}\,(h-z)\phi'(x); \quad \sigma_{yz}^{(2)} = \frac{1}{\lambda_1}(h-z)\psi'(x)\,; \quad \sigma_{zz}^{(2)} = \frac{1}{\lambda_1}\frac{(h-z)^2}{2}\phi''(x)$$

where $\phi(x)$, $\psi(x)$ and $\eta(x)$ are unknown functions of CL1.In this paper, polynomial series with unknown degrees of freedom (DOF) were used for these functions. Thus, based on Eq.13, they can be expressed as follows:

$$\phi(x) = \sigma_{xx}^{0(1)} * \left(1 - \frac{\left(x^2 - a^2\right)^2}{a^2}\right) * \sum_{i=1}^{m} d[i] * \left(\frac{x}{a}\right)^{2(i-1)} \tag{17}$$
$$\psi(x) = \sigma_{xy}^{0(1)} * \left(1 - \frac{\left(x^2 - a^2\right)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)}$$
$$\eta(x) = \sum_{i=2m+1}^{3m} d[i] * \left(\frac{x}{a}\right)^{2(i-2m+1)}$$

where the *d*-vector contains the unknown parameters (DOF) of the polynomial series and “*m”* is number of DOF for each function.

2.2. External cracks

The same procedure but with different boundary conditions could be followed for laminates with external crack (Fig. 2.(b)). For this case, Eq.5 is defined as:

$$\sigma_{xx}^{(2)}(\pm a, z) = -\sigma_{xx}^{0(2)}(x); \qquad \sigma_{xy}^{(2)}(\pm a, z) = -\sigma_{xy}^{0(2)}(x)\,; \ \sigma_{yz}^{(2)}(\pm a, z) = 0; \tag{18}$$

Considering Eq.6 to 8 as well as Eq.14 and 18, stress functions for HL1 can be found as:

$$\sigma_{xx}^{(1)}(x) = \lambda_1\,\phi(x); \qquad \sigma_{xy}^{(1)}(x) = \lambda_1\,\psi(x); \qquad \sigma_{yy}^{(1)}(x) = \lambda_1\,\eta(x)$$
$$\sigma_{xz}^{(1)} = \phi'(x)\,z; \qquad \sigma_{yz}^{(1)} = \psi'(x)\,z; \qquad \sigma_{zz}^{(1)} = \frac{(ht_1 - z^2)}{2}\phi''(x) \tag{19}$$

and for external cracked layer (CL2), one can find stress functions as below:

$$\sigma_{xx}^{(2)}(x) = -\phi(x); \quad \sigma_{xy}^{(2)} = -\psi(x); \quad \sigma_{yy}^{(2)}(x) = -\eta(x) \quad (20)$$

$$\sigma_{xz}^{(2)} = -(h-z)\phi'(x); \quad \sigma_{yz}^{(2)} = -(h-z)\psi'(x); \quad \sigma_{zz}^{(2)} = -\frac{(h-z)^2}{2}\phi''(x)$$

where $\phi(x)$, $\psi(x)$ and $\eta(x)$ are unknown function of exterior lamina. Thus, considering new boundary conditions, they can be expressed as follows:

$$\phi(x) = \sigma_{xx}^{0(2)} * \left(1 - \frac{\left(x^2-a^2\right)^2}{a^2}\right) * \sum_{i=1}^{m} d[i] * \left(\frac{x}{a}\right)^{2(i-1)} \quad (21)$$

$$\psi(x) = \sigma_{xy}^{0(2)} * \left(1 - \frac{\left(x^2-a^2\right)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)}$$

$$\eta(x) = \sum_{i=2m+1}^{3m} d[i] * \left(\frac{x}{a}\right)^{2(i-2m+1)}$$

2.3. Internal cracks

Two homogenized laminate (HL1 and HL2) at the top and bottom of cracked lamina (CL3) should be considered for laminated containing internal matrix crack (*Fig. 3*.(c)). So, According to Eq. 11 and 12, two series of stress functions are required to estimate stress distribution of the cracked laminate.

For this case, Eq.5 is defined as:

$$\sigma_{xx}^{(3)}(\pm a, z) = -\sigma_{xx}^{0(3)}(x); \quad \sigma_{xy}^{(3)}(\pm a, z) = -\sigma_{xy}^{0(3)}(x); \quad \sigma_{xz}^{(3)}(\pm a, z) = 0; \quad (22)$$

Following Eq.6 to 8, stress functions for CL3 can be found as below.

$$\sigma_{xx}^{(3)}(x) = -\phi_3(x); \quad \sigma_{xy}^{(3)}(x) = -\psi_3(x); \quad \sigma_{yy}^{(3)}(x) = -\eta_3(x);$$

$$\sigma_{xz}^{(3)}(x) = t_1\left(\left(\frac{z}{t_1} - (1+\lambda_2)\right)\phi_3'(x) - \lambda_1\phi_2'(x)\right); \quad \sigma_{yz}^{(3)}(x) = t_1\left(\left(\frac{z}{t_1} - (1+\lambda_2)\right)\psi_3'(x) - \lambda_1\psi'(x)\right)$$

$$\sigma_{zz}^{(3)}(z) = \left(\frac{\lambda_1 t_1^2}{2}\right)\left(2\left(\frac{z}{t_1} - 1\right) - (\lambda_1 + 2\lambda_2)\right)\phi_2''(x) + \left(-\frac{z^2}{2t_1} + (1+\lambda_2)z - t_1(1+\lambda_2)^2\right)\phi_3''(x)$$

$$\lambda_2 = \frac{t_3}{t_1} \quad (23)$$

Similarly, for upper homogenized laminas (HL2), stress functions can be found

$$\sigma_{xx}^{(2)}(x) = -\phi_2(x); \quad \sigma_{xy}^{(2)}(x) = -\psi_2(x); \quad \sigma_{yy}^{(2)}(x) = -\eta_2(x); \quad (24)$$

$$\sigma_{xz}^{(2)}(x) = -(h-z)\phi_2'(x); \qquad \sigma_{yz}^{(2)}(x) = -(h-z)\psi_2'(x)$$

$$\sigma_{zz}^{(2)}(z) = \left(hz - \frac{z^2}{2} - \frac{h^2}{2}\right)\phi_2''(x)$$

Finally, Eq.11 and 12 leads to the stress functions for HL1 as below.

$$\sigma_{xx}^{(1)}(x) = \lambda_1\phi_2(x) + \lambda_2\phi_3(x); \qquad \sigma_{xy}^{(1)}(x) = \lambda_1\psi_2(x) + \lambda_2\psi_3(x);$$
$$\sigma_{yy}^{(1)}(x) = \lambda_1\eta_2(x) + \lambda_2\eta_3(x);$$
$$\sigma_{xz}^{(1)}(x) = -(\lambda_1\phi_2'(x) + \lambda_2\phi_3'(x))z; \qquad \sigma_{yz}^{(1)}(x) = -(\lambda_1\psi_2'(x) + \lambda_2\psi_3'(x))z \tag{25}$$

$$\sigma_{zz}^{(1)}(z) = \left(\frac{\lambda_1 t_1^2}{2}\right)\left(\left(\frac{1}{t_1^2}\right)z^2 + (1+\lambda_1+2\lambda_2)\right)\phi_2''(x) + \left(\frac{t_1^2}{2}\right)\left(\lambda_2\left(\frac{z}{t_1^2} - \lambda_2\right) + 2\lambda_1\right)\phi_3''(x) \tag{26}$$

Thus, polynomial stress functions for CL3 and HL2 can be defined as below

$$\phi_3(x) = \sigma_{xx}^{0(3)} * \left(1 - \frac{\left(x^2-a^2\right)^2}{a^2}\right) * \sum_{i=1}^{m} d[i] * \left(\frac{x}{a}\right)^{2(i-1)} \tag{27}$$
$$\psi_3(x) = \sigma_{xy}^{0(3)} * \left(1 - \frac{\left(x^2-a^2\right)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)}$$
$$\eta_3(x) = \sum_{i=2m+1}^{3m} d[i] * \left(\frac{x}{a}\right)^{2(i-2m+1)}$$

$$\phi_2(x) = \sum_{i=1}^{m'} d[i] * \left(\frac{x}{a}\right)^{2(i-1)} \tag{28}$$
$$\psi_2(x) = \sum_{i=m'+1}^{2m'} d[i] * \left(\frac{x}{a}\right)^{2(i-m'+1)}$$
$$\eta_2(x) = \sum_{i=2m'+1}^{3m'} d[i] * \left(\frac{x}{a}\right)^{2(i-2m'+1)}$$

3. Complementary Energy

For a cracked lamina, the complementary energy is the sum of the complementary energy of un-cracked laminate and complementary energy due to stress perturbation [22]. Thus, the complementary energy is:

$$U = U_c^0 + U'_c \tag{29}$$

where:

$$U_{0c} = \frac{1}{2}\int \sigma^0 S^0 \sigma^0 dV = \frac{1}{2}\bar{\sigma} S^0 \bar{\sigma}\ V \quad ; \qquad U'_c = \frac{1}{2}\int \sigma' S \sigma' dV \tag{30}$$

and S is the local compliance matrix, S^0 is effective compliance matrix, and V is the volume of the laminate, and $\bar{\sigma}$ and σ' are average stress tensors equal to the applied stress and perturbation stress, respectively.

Based on the stress transformation rule in laminated theory, the stress tensor in local coordinate system of lamina-i can be written as follows:

$$(\sigma^0)^{(i)} = \bar{\sigma}^T\, S^0\, \bar{Q}^{(i)}\, T^T \tag{31}$$

where $\bar{Q}^{(i)}$ is the stiffness matrix of the i-lamina in cracked coordinate system and T is stress rotation matrix corresponding to the rotation angle ($\theta^{(k)}$-90). Based on the principle of minimum complementary energy for a cracked body:

$$U = \frac{1}{2}\,\bar{\sigma}\, S^*\, \bar{\sigma}\, V \leq \frac{1}{2}\,\bar{\sigma}\, S^0\, \bar{\sigma}\, V + \frac{1}{2}\int \sigma' S\, \sigma'\ dV \tag{32}$$

where, S^* is effective compliance matrix of the cracked laminate. Thus, minimization of the right-hand side of the above inequality, the upper bond of reduced compliance matrix and stiffness reduction can be calculated as follows:

$$D(i) = 1 - \frac{S[i,i]_{\ perturbation} + S^0(i,i)}{S^0(i,i)} \tag{33}$$

where:

$$S[i,i]_{\ perturbation} = \frac{2U'_c}{(\bar{\sigma})^2 Vol} \tag{34}$$

$$S^0 = 2\, H\, A^{-1} \tag{35}$$

where A is the extensional stiffness matrix of the laminate and H is the total thickness of the laminate. Note that, in order to calculate the complementary energy due to stress perturbation, the complementary energy of each lamina must be calculated as follows:

$$U'_c = \int_{-a}^{a}\int_{0}^{t1} W^1\, dz\, dx + \int_{-a}^{a}\int_{t1}^{h} W^2\, dz\, dx \tag{36}$$

where, W^i is the perturbation stress energy density of the lamina(i) in cracked coordinate system:

$$W^1 = \frac{1}{2}\,\sigma'^{(i)} S^{(i)}\, \sigma'^{(i)} \tag{37}$$

In this approach, [σ'] is a series of polynomial functions with unknown DOFs, based on (2), (3) and (17). For instance, the stress perturbation for lamina (1) with a center cracked laminate is:

$$
\{\sigma\}_x{}^{(1)} = \begin{bmatrix} -\sigma_{xx}^{0(1)} * \left(1 - \frac{(x^2-a^2)^2}{a^2}\right) * \sum_{i=1}^{m} d[i] * \left(\frac{x}{a}\right)^{2(i-1)} \\ \sum_{i=2m+1}^{3m} d[i] * \left(\frac{x}{a}\right)^{2(i-2m+1)} \\ \frac{[h\,t_1 - z]^2}{2} \frac{\partial^2}{\partial x} \left(\sigma_{xx}^{0(1)} * \left(1 - \frac{(x^2-a^2)^2}{a^2}\right) * \sum_{i=1}^{m} d[i] * \left(\frac{x}{a}\right)^{2(i-1)}\right) \\ \frac{\partial}{\partial x}\left(\sigma_{xy}^{0(1)} * \left(1 - \frac{(x^2-a^2)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)}\right) Z \\ \frac{\partial}{\partial x}\left(\sigma_{xy}^{0(1)} * \left(1 - \frac{(x^2-a^2)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)}\right) Z \\ -\sigma_{xy}^{0(1)} * \left(1 - \frac{(x^2-a^2)}{a^2}\right) * \sum_{i=m+1}^{2m} d[i] * \left(\frac{x}{a}\right)^{2(i-m+1)} \end{bmatrix} \quad (38)
$$

and finally, U'_c could be written based on unknown DOFs and Newton-Raphson procedure is applied to minimize

$$
U'_c = f\big(\phi(x)\,\psi(x)\,\eta(x)\big) = f(\,d_1, d_2, \dots d_{3m}) \quad (39)
$$

3.1. Numerical Procedure

By adding the stress perturbation into the complementary energy and applying the minimum complementary energy principle, series of nonlinear equations are generated that must be solved numerically in order to find unknown parameters of stress perturbations (17), (21), (27) and (28). For this purpose, the Newton-Raphson iterative procedure is utilized to obtain the value of parameters (DOF) and the stress field as follows:

$$
[K_d] = \frac{\partial U'_c}{\partial d_i} \quad (40)
$$

where [K_d] is the differentiation of the total complementary energy and [d] is the vector of DOF. Newton-Raphson procedure is defined as Eq. (18):

$$
\begin{aligned} [R_{\,i+1}] &= [R_{\,i}] + [K_d\,(d_{\,i})] \\ [d_{\,i+1}] &= [d_{\,i}] - [K_T(d_{\,i})]^{-1} \times [R_{\,i+1}] \end{aligned} \quad (41)
$$

where K_T is tangential stiffness matrix defined as:

$$
[K_T] = \frac{\partial^2 U'_c}{\partial d_i\,\partial d_j} \quad (42)
$$

and [R] is the residual vector. The iteration is terminated when the L2 norm ||R||:

$$
\| R \| = \sqrt{\sum_{i=1}^{m} R_i^{\,2}} \quad (43)
$$

of the residual is sufficiently small, namely smaller than an arbitrary tolerance value δ. Finally, once the values of the m parameters are known, the stress perturbations are calculated using (5). Even is

||R||<delta, the solution may be inaccurate if the number of terms “m” in the series is not large enough. This is addressed in section 4.1.

Although not the purpose of this work, $[K_T]$ and $[K_d]$ could be solved just one time by a symbolic calculation over the geometry parameters, material properties and loading conditions (i.e. crack density, lamina angle, Young modulus). So, it can be embedded in FE software package to be used for any values of parameters. For instance, for a thin center cracked laminate with $m = 2$, K_d [3] is as follow:

$$K_d[3] = {\sigma_{0\,xy}^{(1)}}^2 \left(1.1\, S_1(6,6)\ a\, t_1 + \frac{0.89\, S_1(4,4)\, t_1^3}{a} + \frac{0.89\, S_2(4,4)\, t_2^3}{\lambda_1^2\, a} + \frac{1.1\, S_2(6,6)\, a\, t_2}{\lambda_1^2}\right) dm[3] + {\sigma_{0\,xy}^{(1)}}^2 \left(0.15\, S_1(6,6)\ a\, t_1 + \frac{0.18\, S_1(4,4)\, t_1^3}{a} + \frac{0.18\, S_2(4,4)\, t_1^3}{\lambda_1^2 a} + \frac{0.18\, S_2(6,6)\, a\, t_2}{\lambda_1^2}\right) dm[4] + {\sigma_{0\,xy}^{(1)}}^2 \left(1.3\, S_1(6,6) a\, t_1 + \frac{1.3\, S_2(6,6) a\, t_2}{\lambda_1^2}\right) \quad (44)$$

where Si is the compliance matrix of HL1 at cracked coordinate system.

4. Results and Discussion

In order to verify the proposed method, several comparisons between experimental and analytical results using different material properties are performed. *Fig. 3* shows the obtained non-dimensional shear stress perturbation for a $[0/90]_s$ Graphite/Epoxy laminate under pure shear loading with Material #1 given in *Table 1*, and crack density ρ=1.96 [1/mm]. The results are compared in with those obtained by analytical variational approach in [57] and it shows acceptable consistency between the two results series.

Fig. 4 compares the obtained longitudinal stiffness reduction of $[\theta/-\theta]_s$ for θ = 15, 30, 45, 60, 75 with those available in [55] with material properties of Material #2 in Table 1. It can be seen that the method is capable to predict the stiffness reduction of the laminates at small and large crack densities, acceptably. Moreover, by increasing the value of θ, the stiffness reduction is more pronounced. Similar results, but for laminate under in-plane shear loading, are illustrated in *Fig. 5* . It is clear that increasing the crack density has less effect on the shear modulus reduction for $[45/-45]_s$ laminate under in-plane shear loading comparing with other angle-ply laminates.

Note that longitudinal stiffness (*Fig. 4*) and shear stiffness (*Fig. 5*) asymptotically approach values larger than zero for all angles in all cases. It is thus clear that cohesive zone models [Abaqus PDA, ANSYS PDA], that assume zero stress for infinite crack density, are unrealistic.

Table 1 Material Properties

Material	E_A (MPa)	E_T (Mpa)	G_A (Mpa)	G_T (MPa)	ν_A	ν_T	t (mm)
#1 [57]	130000	9720	5390	3360	0.31	0.49	0.127
#2 [55]	43000	13000	3400	4580	0.30	0.420	0.203
#3 [6]	135000	11000	5800	4200	0.301	0.45	0.124

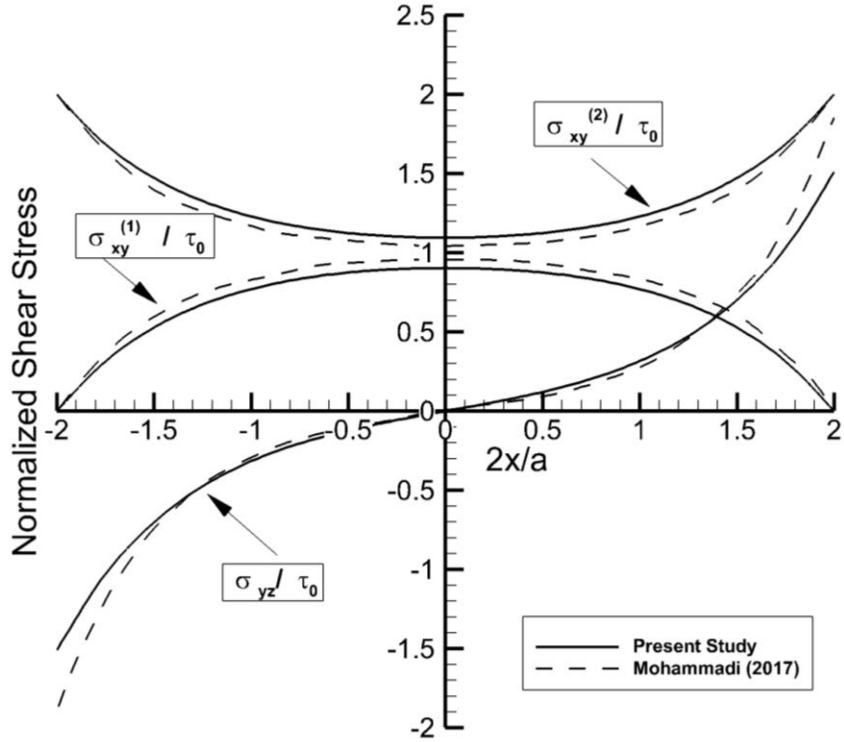


Fig. 3 Comparison of the obtained stress perturbation for $[0/90]_s$ laminate under pure shear stress loading with results in [57] (crack density ρ = 1.96 1/mm)

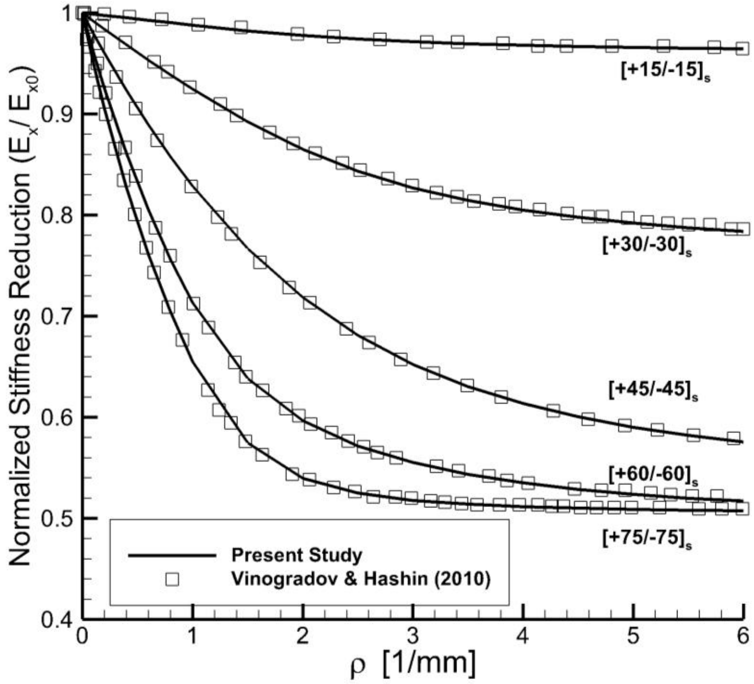


Fig. 4 Comparison of the obtained longitudinal stiffness reduction versus crack density for several angle-ply laminates with those available in [55]

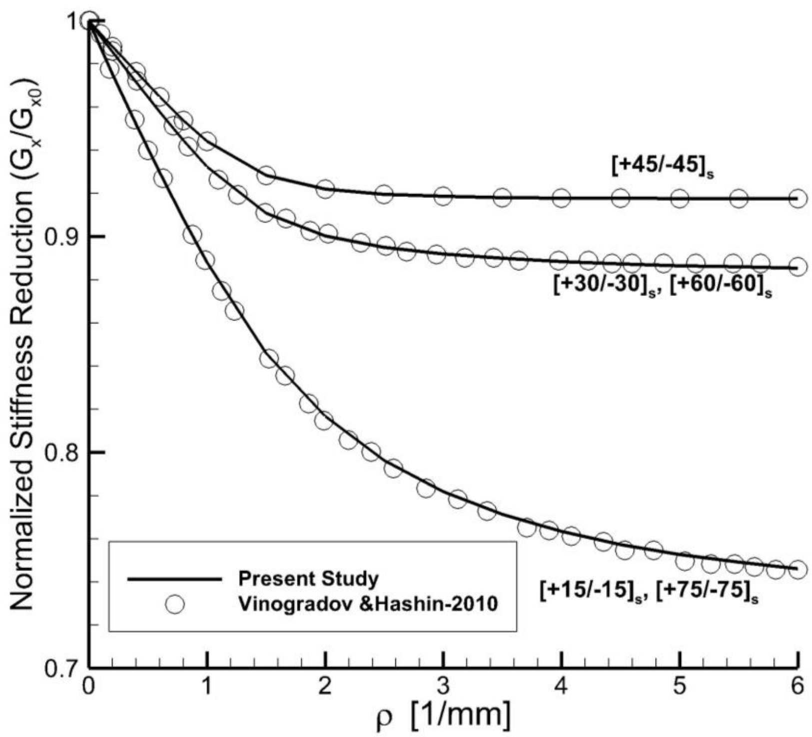


Fig. 5 Comparison of the obtained stiffness reduction versus crack density for several angle-ply laminates under in-plane shear loading with those available in [55]

The normalized stiffness reduction of $[0_2/55_2]_s$ laminate with material #3 reported in Fig. 6 are compared to those available from [6].The results from this work are below those from [15] because the variational approach are a lower bound of stiffness. It is also evident that the effect of crack density on the longitudinal stiffness reduction of the $[0_2/55_2]_s$ laminate is negligible while it has a major effect on the transverse and shear modulus reductions. Fig. 7 compares the results of normalized longitudinal stiffness reduction versus crack density of $[+45^{\circ}]$ for a $[\pm 45]_s$ with those available in [15, 61]. Note that, the value of crack density for $[-45^{\circ}]$ is equal to1.0 [1/mm]. Therefore, stiffness reduction due to matrix cracking at $[-45^{\circ}]$ is first calculated, then the homogenized reduced stiffness accounts for calculation of stiffness reduction at $[+45^{\circ}]$. Agreement between result shows that the developed method is able to predict stiffness reduction for both middle and external cracks very well.

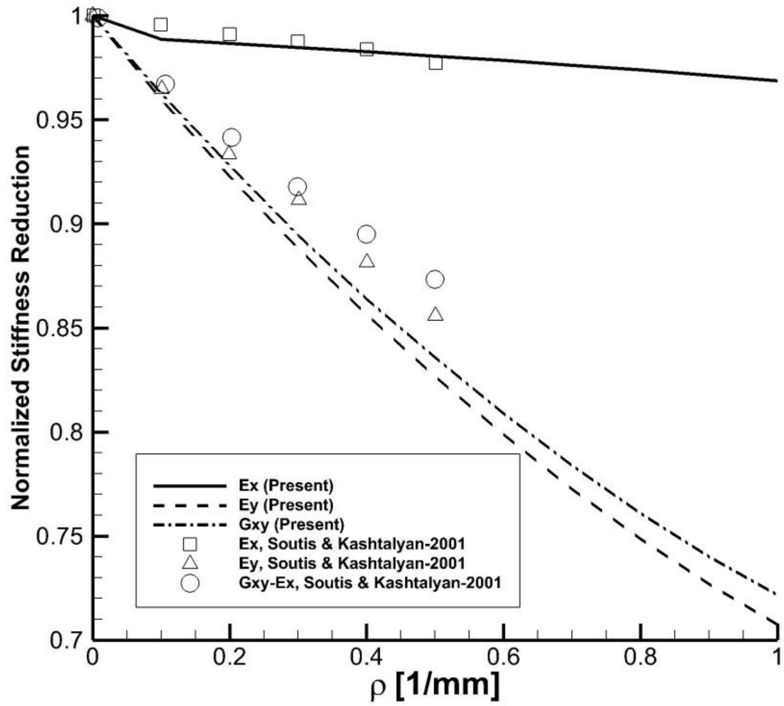


Fig. 6 Comparison of obtained normalized stiffness reduction versus crack density for $[0_2/55_2]_s$ laminate with those available in [6]

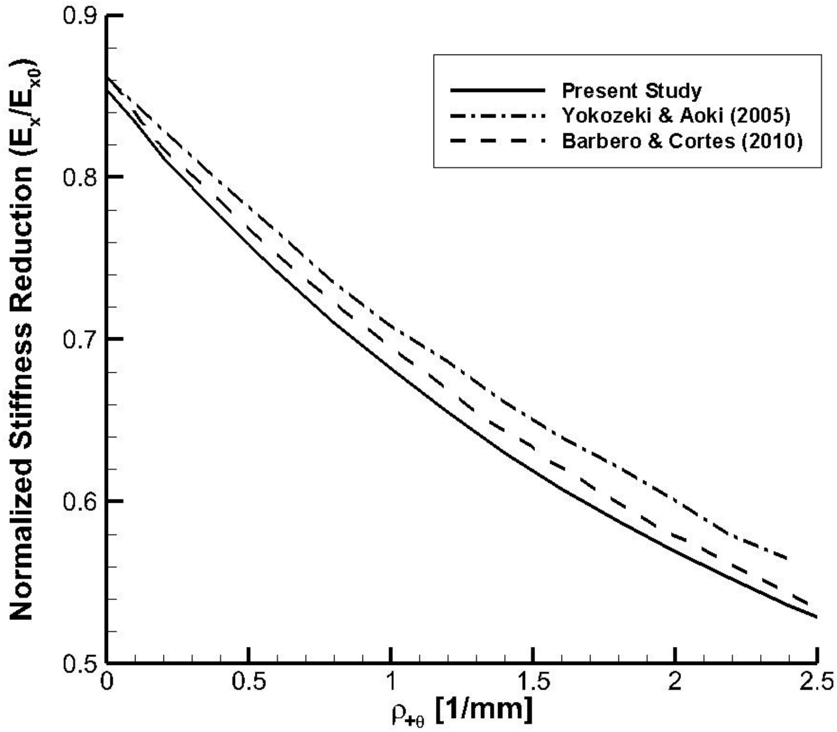


Fig. 7 Comparison of obtained normalized stiffness reduction versus crack density ($\rho_{(+\theta)}$) for $[\pm 45]_s$ laminate with $\rho_{(-\theta)}$ = 1.0 [1/mm] with those available in [15, 61]

Fig. 8 and Fig. 9 compare stiffness reduction of two middle cracked laminates, $[0/90_8/0_{0.5}]_s$ and $[0/55_4/\text{-}55_4/0_{0.5}]_s$ with available experimental [62] and also analytical [48] data. It can be seen that the presented method is also able to predict well the stiffness reduction of internal cracked laminates.

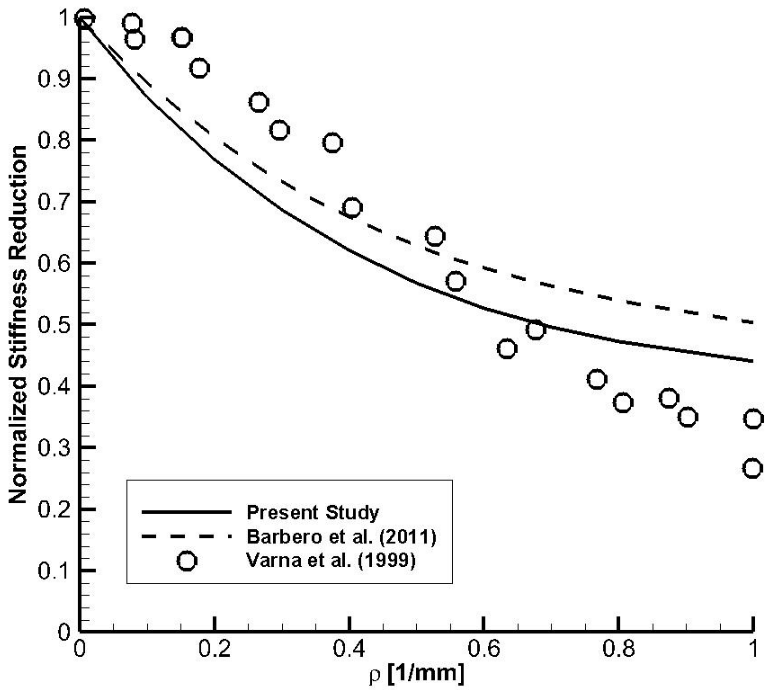


Fig. 8 Comparison of Normalized stiffness reduction for Glass/Epoxy $[0/90_8/0_{0.5}]_s$ with those availabe in [48, 62]

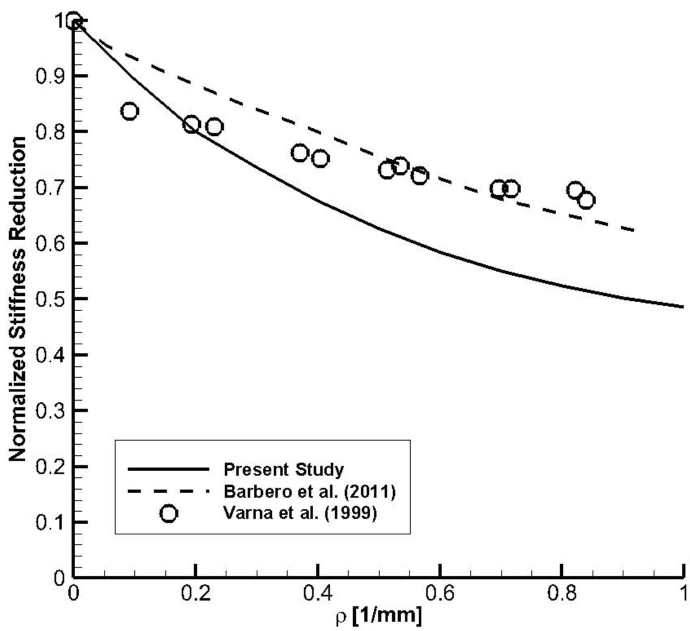


Fig. 9 Comparison of Normalized stiffness reduction for Glass/Epoxy $[0/55_4/\text{-}55_4/0_{0.5}]_s$ with those availabe in [48, 62]

Although numerical approaches such as proposed in this work are advantageous due to generality (stacking sequences, boundary conditions and etc.), they are computationally expensive when compared to analytical solutions [50]. Thus, reduction of computational time is important. *Fig. 10* shows the stress perturbation of 90° lamina for a $[0/90]_s$ laminate with material #1 for different values of crack densities through the length of the lamina. It can be seen that as crack density increases, the perturbation extends to more of the space “2a” between cracks. The polynomials functions used to produce *Fig. 10* are shown in *Table 2*. Based on the presented results in *Fig. 10* and *Table 2*, it is clear that for prediction of ϕ(x) with a crack density of 0.1, a polynomial series function with order of 26 is required. While this polynomial order may decrease with increasing of the crack density value. See Section 4.1.

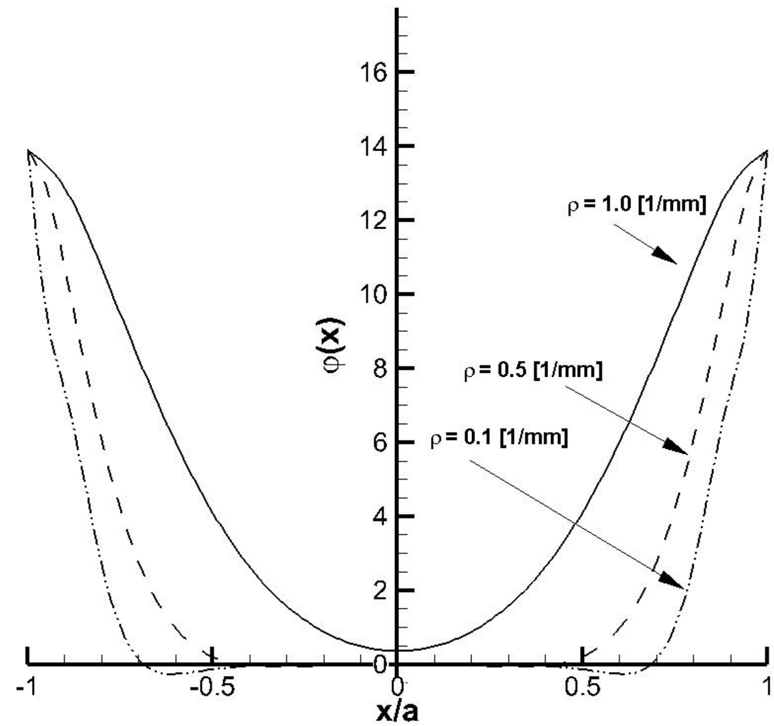


Fig. 10- Longitudinal stress perturbation (ϕ_x) along the length of laminate for a typical cross-ply laminate and different values of crack density, with crack spacing=2a

Table 2- Polynomial series for stress perturbation functions shown in Fig. 10.

$$\boldsymbol{\phi(x)}_{\rho=1.0} = 13.9 - 55.7(x^2 - 0.25)^2(0.0345\,x^{16} + 0.236\,x^{14} + 0.0456\,x^{12} - 3.11\,x^{10} - 7.5\,x^8 + 1.32\,x^6 + 19.2\,x^4 + 16.6\,x^2 + 3.89)$$

$$\boldsymbol{\phi(x)}_{\rho=0.5} = 13.9 - 13.9\,(x^2 - 1.0)^2(0.00514\,x^{16} - 0.0304\,x^{14} - 0.119\,x^{12} - 0.0991\,x^{10} + 0.821\,x^8 + 2.5\,x^6 + 3.02\,x^4 + 2.08\,x^2 + 1.0)$$

$$\boldsymbol{\phi(x)}_{\rho=0.1} = 13.9 - 0.557\,(x^2 - 25.0)^2\,(-1.55e-13\,x^{22} + 1.96e-11\,x^{20} - 9.93e-10\,x^{18} + 2.72e-8\,x^{16} - 4.49e-7\,x^{14} + 4.65e-6\,x^{12} - 3.03e-5\,x^{10} + 1.22e-4\,x^8 - 2.66e-4\,x^6 + 5.08e-4\,x^4 + 0.00306\,x^2 + 0.04)$$

4.1. Sensitivity Analysis

Sensitivity analysis is performed to find the minimum number of DOF, i.e. 3 x *m* in (17), where *m* is the number of DOF for each of three functions (17). *Fig. 11* shows the stiffness reduction of a typical $[0/90]_s$ laminate under tension loading versus the total number of degrees of freedom. The solution is considered to be satisfactory once the stiffness reduction reaches 99.9% of the asymptotic value in Fig. 7. There is an inverse relation between the number of total required DOFs and the crack density value. Moreover, for convergence over the considered range of the crack density parameter ($0.01 < \rho < 1.0$), at least 3m=9 and at most 3m=51 DOF are required.

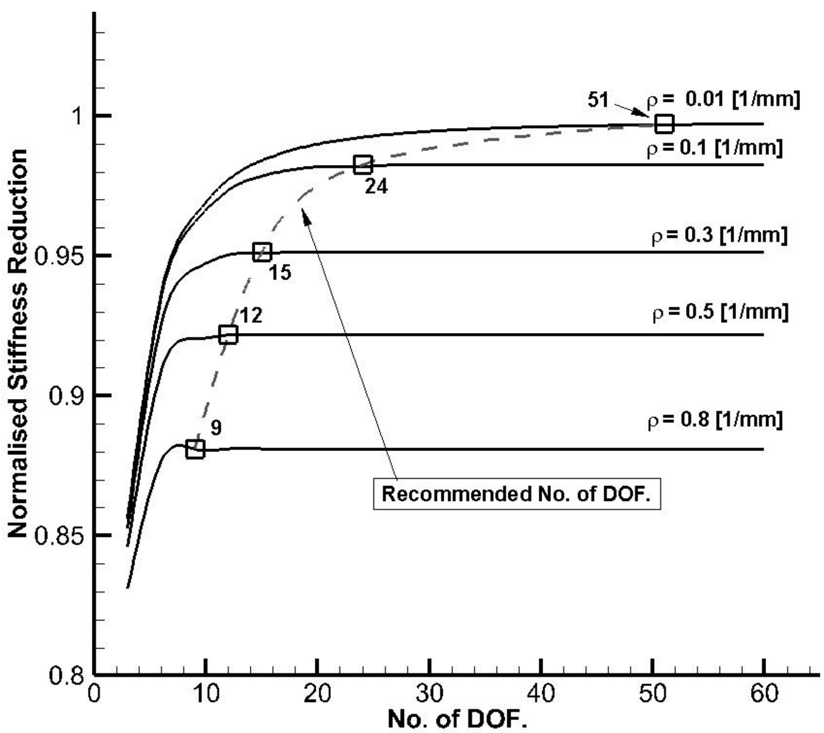


Fig. 11- Sensitivity analysis on number of required 3*m* DOF versus crack density.

Fig. 12 and Fig. 13 show the longitudinal and shear modulus stiffness reductions for $[0/\theta_3]_s$ with material #2. Note that the patterns of stiffness reductions for longitudinal and shear young modulus are not similar for $[0/\theta_3]_s$ laminate with different angles of θ. For instance, although $[0/30_3]_s$ and $[0/90_3]_s$ laminates experience the minimum and maximum stiffness reduction due to matrix cracking, respectively, the trend of the shear modulus stiffness reduction for these laminates is nearly the same. Such conclusion is also true for $[0/45_3]_s$ and $[0/60_3]_s$ laminates.

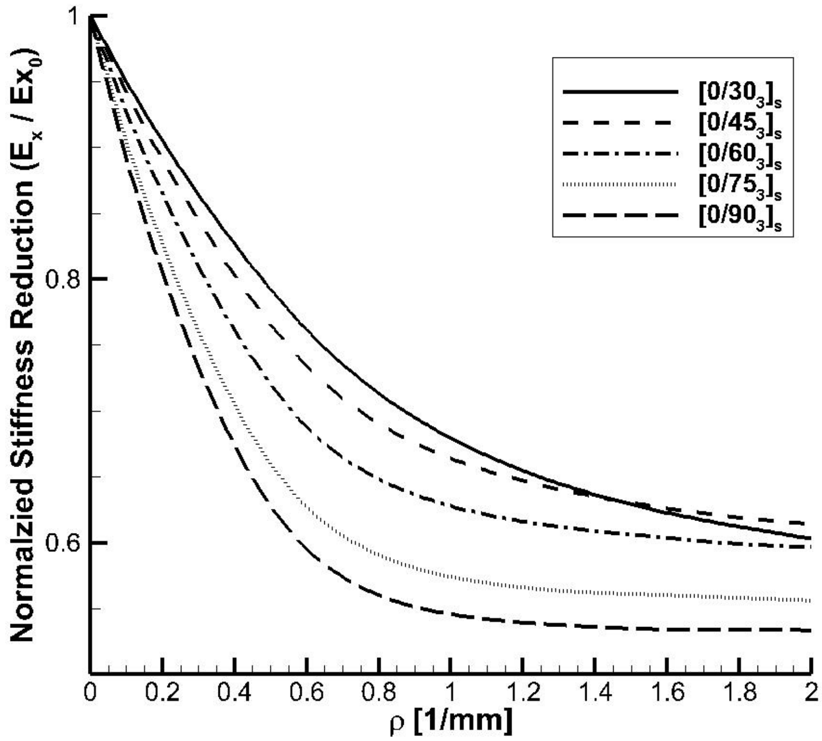


Fig. 12- Normalized longitudinal stiffness reduction (E_x/E_{x0}) versus crack density for $[0/\ \theta]_s$ laminates.

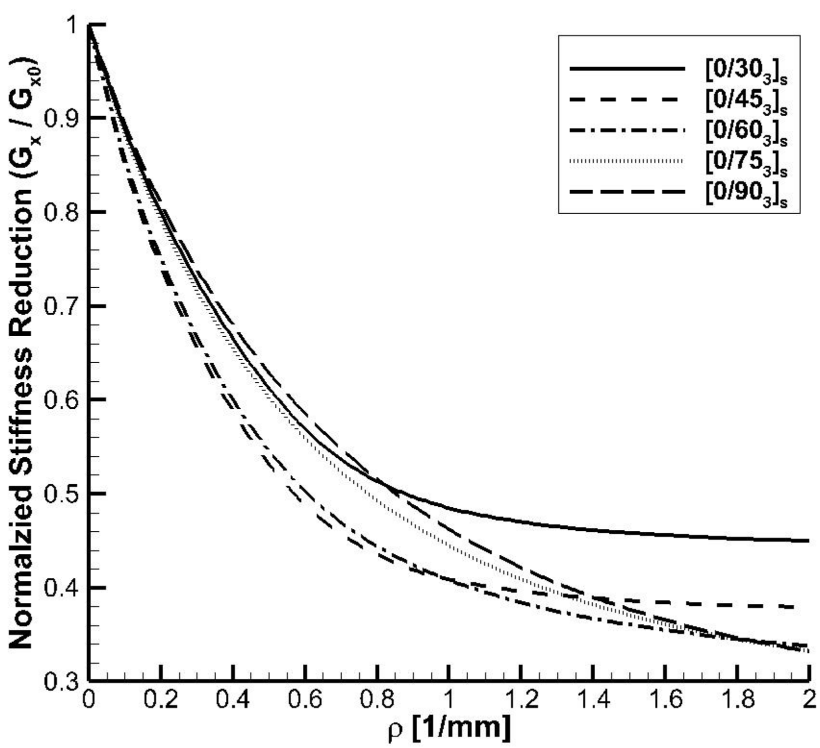


Fig. 13- Normalized shear modulus stiffness reduction (G_x/G_{x0}) versus crack density for $[0/\ \theta]_s$ laminates.

5. Conclusions

In this study, a variational approach is developed for calculating the stiffness reduction of general symmetric laminates containing matrix cracks in both internal and external layers. The main advantage of the proposed method is to take advantage of the benefits of analytical and numerical methods working together. Specifically, polynomial series are used to approximate the stress perturbation analytically (with a series) and their coefficients are calculated by minimizing the complementary energy. Furthermore, the analytical part could be solved once as a function of general parameters (i.e. material properties, staking sequence, and crack density; Section 3.1) for potential incorporation into FEM codes. Stiffness reduction due to matrix cracking of several angle-ply laminates are compared with experimental, numerical, and semi-analytical approaches and results show good agreement. Furthermore, in order to reduce computational time, a sensitivity analysis study on the number of DOF for the proposed method is carried out, and the minimum required number of DOFs for different values of crack densities is recommended. Further work could deal with the generalization of this approach for general unsymmetric laminates.

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