

# Matrix Cracking in Non-symmetric Laminates under Combined Membrane and Flexural Loading

A. Adumitroaie, E. J. Barbero, and M. Schagerl

**Abstract**—A constitutive model of progressive matrix cracking in fiber reinforced laminates is developed for the case of both membrane and flexural deformation. The analytical progressive damage model addresses both the degraded mechanical properties of the laminate for given levels of matrix cracking in individual plies, and the damage growth under applied loading.

Crack densities in individual plies are the damage state variables of the model. This formulation is unlike the progressive damage models for composites implemented in most of the FEA packages, where softening laws are considered in order to describe stiffness reduction and damage evolution. By using the ply crack densities as state variables the model is able to predict and to keep track of the crack density in individual plies during the loading history, which can be of interest in application where the permeability (leakage) of the laminate is a limiting design factor. One example can be pressure vessels containing fluids or gases. Thermal residual stresses are taken into account in the present model, which extends the predictive capabilities of the model to applications in the range of cryogenic temperatures.

The analytical model is validated against available experimental data for the case of both membrane and flexural loading.

**Index Terms**—Composite material, matrix cracking, progressive damage.

## I. INTRODUCTION

The characterization of composites has been an intensive field of research since their emergence as a new class of high performance structural materials. On one hand, the advantages offered by composite materials are very tempting, opening a large pool of choices for both new applications and alternative solutions for traditional applications, e.g., lightweight structures. On the other hand, the use of composite materials requires knowledge of limiting conditions under lifetime loading for each application, and this knowledge requires extensive amount of research. As a matter of fact, the same anisotropy and heterogeneity of the material properties that governs their unique behavior, also raises the complexity in quantifying and understanding the behavior and the properties. Thus, new experimental and analytical tools are constantly evolving to study composite materials.

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Damage mechanisms such as fiber-matrix debonding, fiber breaking, fiber micro-buckling, matrix cracking, inter-ply delamination, follow different laws particular of the material structure and loading case. These modes can be catastrophic (i.e., structural failure) or not, but regardless of their primary structural impact, each one has to be very well understood and controlled. All of them can cause deterioration in the mechanical property and structural integrity, and in some cases they interact, as it is the case when a non-catastrophic damage mode may work as a nucleation point of a catastrophic damage mode.

It is also the case that *failure* is defined not only considering the ultimate structural load carrying capacity, but also considering the functionality of the structure. In this way, some damage mode that is usually considered as non-catastrophic, e.g. matrix cracking can become a limiting one. This case can be best exemplified for the case of a pressure vessel: failure might be defined when the first crack appears and the structure stops meeting its functional requirements, i.e., containing the pressurized fluid or gas. Thus, matrix cracking can be regarded as a limiting mode of damage in some applications where it can constitute a path way for gas or liquid penetration, leading to exterior leakage or even the more difficult to detect fibers corrosion. The latter may induce weakening of fibers and ultimately fiber breakage.

The matrix cracking damage mode in laminated fiber reinforced polymers (FRP) has been studied extensively by the composite research community. The first analytical models were developed for the classical example of the  $[0_n/90_n]_S$  cross-ply laminate [1]-[3], featuring cracks in the transverse central  $90^\circ$  plies. Then, the study was extended to more general cases, like:  $[S/90_n]_S$  laminates [3]-[6], where the letter *S* denotes a *sub-laminate* (not necessarily  $0^\circ$  orientation) that does not undergo matrix cracking under the considered loading case;  $[S/90/S/90/. . . /S/90]_S$  laminates, featuring matrix cracking in multiple transverse  $90^\circ$  plies;  $[S/\theta_1/\theta_2]_S$  laminates [7]-[9], where the loading case is such that matrix cracking develops in the two off-axis  $\theta$  plies;  $[\pm\theta/90_n]_S$  laminates [10], [11], where matrix cracking can develop in both  $\theta$  and  $90^\circ$  plies;  $[\theta_1/\theta_2/. . . /\theta_n]_S$  laminates [12], [13], where matrix cracking can appear in all off-axis  $\theta$  plies. A common limiting characteristic of these models is the fact that the laminate has to be symmetric, and to preserve the symmetry after damage onset. This constrain on the laminate configuration brings, in turn, a constrain on the loading case: flexural loading cannot be addressed, since in that case matrix cracking appears only in those plies loaded in tension, and the symmetry of the laminate is lost. Thus, most of the existing models in the literature deal only with *symmetric*

*laminates* under *in-plane* loading. Very often, even this case is further constrained to uni-axial loading, in order to avoid generating damage in the aforementioned *S* sub-laminates.

Models considering the flexural deformation are even rarer in the literature, and, in most of the cases, considering bending deformation brings additional restrictions on the level of generality and applicability of the solution. Again, only particular LSS can be addressed; or, due to some build-in modeling assumptions, all damaged plies must have the same matrix crack density, condition which cannot be satisfied if there are multiple cracking plies under flexural loading.

Numerical solutions, especially the FEA method, can be an approach alternative to analytical models, for solving the complex 3D stress-strain fields, of the matrix cracking in laminated composites [14]-[16]. Theoretically it is possible to solve the problem of the reduced material properties by FEA method, but these detailed analyzes apply to particular laminate configurations. A parametric study using FEA is difficult to implement, involving multiple meshing. We also have to remember that a real problem, of a real structure considering loading, material properties and boundary conditions as required by the FEA method, becomes a double FEM problem. First of all, the whole structure is discretized and meshed to find the FEA solution for the considered loading case and boundary conditions. Then, when the resulting nonuniform stress field locally reaches the level of damage growth, a second FEA problem has to be solved, to provide the new (reduced) material properties at the laminate scale, for the corresponding extent of damage at the local (ply and material point) level. The double FEM problem continues for each of the external loading increments. The double meshing process becomes very cumbersome and expensive computationally. An easier way to solve the problem [17] consists of having an analytical solution for the problem of reduced material properties at a certain damage level (*reduced material model*), and for the problem of damage extent at a certain loading level (*damage growth model*). The analytical solution is then provided to the FEA problem of the real structure under loading and boundary conditions: the material properties of the individual finite element integration points are modified according to the analytical solution of the *reduced material model* and *damage growth model*.

The objective of the work presented here is the development of an analytical model for matrix cracking progressive damage in composites of general laminate stacking sequence (LSS) under generalized membrane/flexural deformation. Both problems, the *reduced material model* and the *damage growth model* are addressed. There is no limitation on the configuration of the laminate or on the number of the cracking plies. Various other aspects that can influence the matrix cracking process are considered and included in the present model: the effect of thermal residual stress; the possible fracture modes *I*, *II*, or mixed *I-II*; the possibility of crack closure under flexural deformation; the experimentally observed strengthening of the material (i.e., increase resistance of the material to damage growth) with increasing damage level, usually referred to as *Resistance-curve (R-curve)* behavior; the *in-situ* constraining

influence of the neighboring plies on the progressive damage of the cracking ply, and the influence of the thickness of the cracking ply on damage initiation and progression.

The proposed progressive damage model makes use of the following key ingredients: i) an appropriate *COD*- based reduced material model [14], [18]; ii) an energy based damage evolution criterion inspired by Fracture Mechanics; iii) an homogenization technique inspired by *CDM*; iv) an iterative procedure in order to detect the conditions for damage growth in individual plies of the laminate, and to increase the damage level when the conditions are met; and v) the Classical Laminate Theory (*CLT*) [19] in order to describe the overall membrane and flexural deformation of the laminated composite.

The model aims to releasing the limiting assumptions used in previous models dealing with the same problem. The resulting analytical model is amenable to easy implementation in a FE formulation, as a nonlinear material behavior, which is able to predict the onset and evolution of matrix cracking damage in laminated composites, and to assess the effect of this damage mode on the material and structural response.

## II. ANALYTICAL MODEL FORMULATION

The area of applicability of the model naturally comes from the main hypotheses under which the model is developed:

- aligned long fiber reinforced laminated composites; matrix cracking takes place in individual plies along the fiber direction;
- linear elastic material behavior up to damage onset; the possible material nonlinearity due to matrix plastic deformation is not considered;
- brittle fracture behavior; there is no energy released for the formation of plastic zones; the progressive damage growth is modeled under the frame of linear elastic fracture mechanics (LEFM);
- propagated cracks; one matrix crack suddenly develops and spans the whole width of the specimen; the progressive damage growth is rather a discrete process characterized by discrete fracture events, than a continuous one;
- there is uniform spatial distribution of the propagated cracks in a ply affected by matrix cracking; cracks are equally spaced between them;
- there is a strengthening effect (R-curve behavior) of the damage growth on the critical parameter, i.e., the critical strain energy release rate (ERR)  $G_c$ , characterizing the material resistance to crack growth;
- the R-curve behavior features an in-situ nature (i.e., ply thickness dependency).

The main ingredients and modules of the model, and the basic principles of how they work and interact together are described in the following:

*Reduced Material Properties* — are approached through the *COD*-based material model in [14]. Here, the overall laminate thermo-elastic properties for a given damage level (crack density) in each ply of the laminate can be calculated; however, no information can be recovered with regard to

individual plies properties (ply reduced stiffness and ply reduced thermal expansion coefficients), which means that the reduced material model in [14] features a *laminated formulation*. For the present model, individual reduced plies properties are needed in order to calculate the strain energy required by the energy based *damage evolution function*. This fact is achieved by the *iterative procedure* of the model.

A *ply homogenization technique* — inspired by the CDM. In this procedure, individual plies having a certain damage level (crack density) are regarded as fictitious homogenized materials having as effective thermo-elastic properties the properties of the cracked ply. Once the reduced ply properties are determined, individual ply cracks are not physically considered, but the ply cracks are rather regarded as being smeared through the volume of the ply.

An *iterative procedure* — meant to detect the damaging ply ( $k_0$  ply in the present notation) at a certain level of loading, according to the *damage evolution criterion*, and to allow the incremental increase in damage in this ( $k_0$ ) ply. Furthermore, after the ( $k_0$ ) ply is given one damage increment, the iterative procedure (also regarded as *LSS iteration* in the following) repeatedly iterates over the stacking sequence of the laminate at the same load level, in order to check for possible multiple increments in damage level of the same damaging ply ( $k_0$ ), or to check for multiple cracking plies at the same load level, due to stress redistribution after the damage level in the initial detected ( $k_0$ ) ply is increased and the ply stiffness is accordingly reduced.

*Damage Evolution* — is approached through a fracture mechanics inspired *Energy Release Rate* criterion. The damaging ply ( $k_0$ ) is detected by the damage evolution criterion together with the iterative procedure, and the damage level is incrementally increased in this ply.

*Classical Laminated Theory (CLT)* — provides the overall, effective constitutive behavior of the laminate containing cracks in individual plies, provided that the homogenized, effective ply properties are known.

As stated before, two main aspects have to be considered in order to perform *progressive damage analysis* of laminated composites. First, the reduction in the material thermo-elastic constants at a given damage level has to be evaluated; this aspect is regarded as the *reduced material properties* in the present model, and can be described by the relationship

$$[C] = [C(\lambda^{(k)})] \quad (1)$$

where  $[C]$  generically represents the thermo-elastic properties of the material, and  $\lambda$  represents the state variable accounting for the damage level inside of the material. For the present model of intra-laminar matrix cracking, the damage variable is crack density (i.e., number of cracks per unit length  $\lambda^{(k)} = 1/d^{(k)}$ , where  $d^{(k)}$  is the distance between two consecutive cracks inside of any ply ( $k$ ) of the laminate. The second aspect is the prediction of the damage onset and progression due external thermo-mechanical loads applied to the laminate. This aspect is regarded as *damage growth*, and it can be represented by the relationship

$$\lambda^{(k)} = \lambda^{(k)}(\varepsilon) \quad (2)$$

where  $\varepsilon$  generically represents the thermo-mechanical deformation of the laminate.

### III. LAMINATE REDUCED MATERIAL PROPERTIES

The reduced material model in [14] is implemented in the present progressive damage model. For the *uncracked* material, a stress state  $\sigma_{ij}^{(k)}$ ,  $i, j = 1 \dots 3$  exists in each ply ( $k$ ) of the laminate, for a given deformation state of the laminate. The stress state  $\sigma_{ij}^{(k)}$  results in a corresponding vector  $\tau_i^{(k)}$ ,  $i = 1 \dots 3$  of surface tractions at the *prospective* location of the crack in each ply of the laminate, according to the Cauchy stress resultant formula of classical elasticity:

$$\tau_i = \sigma_{ij} \cdot n_j \quad (3)$$

where  $n_j$  is the vector of the direction cosines of the normal to the crack surface. Thus, the correspondence between the surface tractions  $\tau_i$  and the fracture mechanics damage modes is:  $i = 1$ -crack opening,  $i = 2$ -crack shearing or  $i = 3$ -crack tearing. Due to the fact that cracks always appear along the fiber direction (i.e., the 1 direction in the c.s. of the ply), the normal direction  $n$  is always along the 2 transverse direction of the cracking ply.

The average crack displacement of cracks in ply ( $k$ ),  $\Delta u_i^{(k)}$ , represents the relative displacement between individual faces of the crack, corresponding to  $i=1,2,3$  modes. The problem of  $\Delta u_i^{(k)}$  is regarded as the superposition of two separate elasticity problems: first the surface tractions  $\tau_i$  on the *prospective* crack surface are calculated for the material regarded as uncracked; then the average crack displacement in ply ( $k$ ) is considered as the result of the linear contribution of surface tractions in all plies of the laminate  $\tau^{(l)}$  ( $l = 1 \dots N$ ), where  $N$  is the number of plies in laminate. In this way, the average crack displacement in ply ( $k$ ) becomes

$$\Delta u_i^{(k)} = t^{(k)} \sum_{l=1}^N \beta_{ij}^{(kl)} \cdot \tau_j^{(l)}, \quad i, j = 1 \dots 3 \quad (4)$$

where  $t^{(k)}$  is the ply thickness. It can be observed that the linear combination in (4) is made through the coefficients  $\beta^{(kl)}$ , which quantify the effect of surface traction in ply ( $l$ ) on the COD in ply ( $k$ ). The coefficients  $\beta_{ij}^{(kl)}$ , also called *COD coefficients*, are function of: (i) LSS configuration, (ii) ply material properties, and (iii) ply crack density (if cracks influence or not each other); they become now the new unknowns of the problem. The physical meaning of the  $\beta_{ij}^{(kl)}$  coefficients is not a direct intuitive one; they act as

compliance between the surface tractions in ply ( $l$ ),  $\tau_j^{(l)}$ , and the resulting COD in ply ( $k$ ),  $\Delta u_i^{(k)}$ , in the linear combination in (4); they are similar to influence coefficients in [20, Appendix].

There is no direct analytical solution for the COD parameters  $\beta_{ij}^{(kl)}$  in (4) for laminated composites. At this point, the approximation of using COD analytical solutions to a closely related, yet different problem, i.e., the fracture mechanics problem of a row of cracks in an infinite/semi-infinite orthotropic plate, is proposed in [21]-[23]. The level of error introduced by this approximation is checked by comparing the output of the analytical model with experimental data and extensive FE simulations on a large variety of laminate types, and a good correlation is found. Moreover, for the cases where the fracture mechanics solution is not available, analytical laws of COD are obtained by extrapolating extensive parametric FE analyzes [22].

Next, an energy balance equation between two equivalent instances of the cracked laminate is applied [14], [18]:

First, the elastic energy  $W_c$  stored in the composite material featuring matrix cracks is calculated as a linear elasticity superposition problem:

$$W_c = W_0 + \Delta W \quad (5)$$

where  $W_0$  is the elastic energy in the uncracked material, and  $\Delta W$  is the change in the elastic energy due to advent of matrix cracks. The change in elastic energy  $\Delta W$  is calculated based on the LEFM Irwin principle (which states that the energy released equals the mechanical work for crack closure) as the mechanical work done by the surface tractions  $\tau_i$  (3) at the *prospective* crack location, through the corresponding COD  $\Delta u_i$  (4).

Second, the cracked laminate is considered in its effective (homogenized) instance, and the same  $W_c$  in (5) is expressed in terms of effective (reduced) material properties  $C_c, \alpha_c$ :

$$W_c = W_c(C_c, \alpha_c) \quad (6)$$

where  $C_c$  and  $\alpha_c$  are the reduced stiffness and the reduced CTE of the cracked laminate, respectively. By identification of the two expressions (5) and (6) of the stored energy  $W_c$  in the damaged material, the reduced thermo-elastic material properties  $C_c, \alpha_c$  can be calculated as closed form analytical solutions:

$$[C_c]_{6 \times 6} = [C_c(\rho^{(k)})], \quad \{\alpha_c\}_{6 \times 1} = \{\alpha_c(\rho^{(k)})\} \quad (7)$$

where  $\rho^{(k)}$  is the normalized crack density in each ply ( $k$ ) ( $k = 1 \dots N$ ) of the laminate. The normalized crack density is

defined as  $\rho^{(k)} = t^{(k)} / d^{(k)} = t^{(k)} \cdot \lambda^{(k)}$ ; the normalized crack density is a measure of ply damage level equivalent to the crack density  $\lambda^{(k)}$  in eq. (1). The  $6 \times 6$  and  $6 \times 1$  subscripts in eq. (7) indicate that the reduced effective (laminate) material properties are for the case of combined membrane and flexural deformation. In classical notation of *CLT*, the reduced stiffness  $C_c$  in eq. (7) is the  $[ABD]$  membrane-flexural stiffness matrix.

This method, based on the concept of COD, provides a convenient closed form solution to the problem of reduction in material properties of the general  $[\theta_1/\theta_2/\dots/\theta_n]_S$  laminate affected by matrix cracks in any of its plies. The main achievements of this approach is that the restriction on the number of cracking plies is released, and the whole set of thermo-elastic constants in eq. (7) can be evaluated, using as only input the uncracked plies properties and the crack density  $\rho^{(k)}$  in each ply of the laminate. A detailed derivation and explanation of the laminate reduced material model can be found in [14], [24].

#### IV. PLY REDUCED MATERIAL PROPERTIES

The thermo-elastic properties of *individual* plies affected by matrix cracking are calculated inside of the *iterative procedure* (or *LSS iteration*). First, at any given load increment, the iterative procedure checks for the cracking ply (noted ( $k_0$ )) inside of the laminate; the cracking ply is identified as being the ply which first meets the condition of the *damage evolution criterion* in eq. (11). When a ( $k_0$ ) cracking ply is found, the crack density level in that ply is incrementally increased at a subsequent value

$$\rho_{i+1}^{(k_0)} = \rho_i^{(k_0)} + \Delta \rho_i \quad (8)$$

Then, the new effective laminate properties are calculated by the *reduced material model* (7). At this point, the iterative procedure provides a means of calculating the reduction in individual properties of the cracking ply ( $k_0$ ): considering that the reduction in effective laminate properties are only due to increasing the crack density in the ( $k_0$ ) ply, the new effective material properties of the ply ( $k_0$ ) are calculated through the *CLT* as

$$\begin{aligned} [Q^{(k_0)}] &= \frac{1}{t^{(k_0)}} \left[ [A^{(k_0)}] - \sum_{k=1}^N (1 - \delta_{k,k_0}) \cdot t^{(k)} \cdot [Q^{(k)}] \right] \\ \{\alpha^{(k_0)}\} &= \frac{1}{t^{(k_0)}} [Q^{(k_0)}]^{-1} \cdot \\ &\cdot \left[ -\{\delta_E^{(k_0)}\} - \sum_{k=1}^N (1 - \delta_{k,k_0}) \cdot t^{(k)} \cdot [Q^{(k)}] \cdot \{\alpha^{(k)}\} \right] \end{aligned} \quad (9)$$

where  $[A^{(k_0)}]$  and  $\{\delta_E^{(k_0)}\}$  are respectively the effective (reduced) laminate in-plane stiffness matrix and the effective laminate in-plane vector of thermal effect coefficients, provided by the material model (7);  $\delta_{k,k_0}$  is the Kronecker-Delta function, which returns the value 1 if  $k = k_0$ ,

or value 0 if  $k \neq k0$ . It has to be noted that the values  $[A^{(k0)}]$ ,  $\{\mathcal{D}_E^{(k0)}\}$  are overall laminate properties; the subscript  $(k0)$  has been used for them only to indicate that they are calculated due to a crack multiplication in  $(k0)$  ply only.

After both effective laminate properties (7), and the effective ply properties (9) are calculated, the iterative procedure is resumed in order to check for other cracking plies at the same level of loading, or for multiple crack increments (8) for the same cracking ply  $(k0)$ . The iteration stops when no ply meets the condition in the damage evolution criterion.

Regarding the increment in crack density  $\Delta\rho$  in eq. (8), different modeling approaches can be considered: an infinitesimal increase in crack density at each cracking event; a discrete increase in crack density corresponding to the formation of one more crack; or a discrete doubling in crack density. The progressive damage model proposed here uses the approach of *1 more crack* discrete crack multiplication process, which provides the incremental value of crack density in eq. (8) [24], [25].

$$\rho_{i+1} = (i + 2)\rho_0 \quad (10)$$

where  $\rho_0$  is the initial starting value of the crack density, corresponding to an arbitrary initially very large crack spacing between preexisting cracks in all plies of the laminate, in the undamaged state.

## V. PROGRESSIVE DAMAGE

A damage evolution criterion is required in order to detect the conditions for matrix crack multiplication in individual plies of the laminate under loading. In the present formulation, a fracture mechanics inspired criterion [26] is selected, in order to consider mixed modes crack formation for the case of matrix cracking under general in-plane and flexural deformation.

$$g_{(k0)} = \left(1 - \frac{G_{Ic}}{G_{IIc}}\right) \sqrt{\frac{G_I^{(k0)}}{G_{Ic}}} + \frac{G_I^{(k0)}}{G_{IIc}} + \frac{G_{II}^{(k0)}}{G_{IIc}} - 1 \geq 0 \quad (11)$$

$G_I^{(k0)}$ ,  $G_{II}^{(k0)}$  are energy release rates due to modes *I*, *II* crack propagation in the ply  $(k0)$  of the laminate. Even if  $G_{I,II}^{(k0)}$  are calculated at global level of the laminate, considering the contribution of all  $(k)$  plies to the energy released due to crack growth in the current  $(k0)$  cracking ply, subscripts  $(k0)$  are used in eq. (11) in order to emphasize that energy release rates are evaluated due to damage multiplication in the current cracking ply  $(k0)$  only.  $G_{Ic}$ ,  $G_{IIc}$  are critical values of the energy release rates, or the material resistance to modes *I* and *II* crack propagation;  $G_{Ic}$  and  $G_{IIc}$  are regarded as material properties whose evaluation is based on experiments.

For displacement control conditions, the strain ERR,  $G$ , is

$$G = -\frac{\Delta U}{\Delta A} \quad (12)$$

where  $\Delta U$  is the strain energy released by the new crack formation, and  $\Delta A$  is the new formed crack area. The strain energy released due to incremental crack multiplication is calculated as the difference between the two instances of the material, *before* and *after* crack growth:

$$\Delta U = U^{after} - U^{before}$$

For the calculation of the strain energy *before/after* crack multiplication, and the partition of the total strain energy in modes *I/II* crack growth [27] as required by the crack growth criterion in eq. (11), the strain energy density stored in the ply  $(k)$  of the laminate is written as

$$U_0^{(k)} = \frac{1}{2} \{\varepsilon^{(k)}\}^T \cdot [\bar{Q}^{(k)}] \cdot \{\varepsilon^{(k)}\} \quad (13)$$

where  $\varepsilon^{(k)}$  is the vector of the strain components of the ply  $(k)$ , written in global (laminate) c.s. However, the  $G_{I/II}^{(k0)}$  in eq. (11) represent the *I/II* modes components of ERR of the laminate due to crack multiplication in the ply  $(k0)$ , where strain energy contribution from all plies  $(k)$  to the crack multiplication in ply  $(k0)$  is taken into account. For this, the  $\varepsilon^{(k)}$  strain of ply  $(k)$  is first transformed to the c.s. of the cracking ply  $(k0)$ , and then separated in components corresponding to the ERR for modes *I* and *II* crack multiplication in ply  $(k0)$

$$\begin{aligned} \varepsilon^{(k)} &= \{\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12}\}^{(k)} \\ &\Rightarrow \begin{cases} \varepsilon_I^{(k)} = \{0, \varepsilon_{22}, 0\}^{(k)} \\ \varepsilon_{II}^{(k)} = \{0, 0, \varepsilon_{12}\}^{(k)} \end{cases} \end{aligned} \quad (14)$$

These mode components are then used for the calculation of strain energy density in eq. (13), and the strain energy of the whole laminate for crack formation in the cracking ply  $(k0)$  is then expressed as the summation of the available strain energy in each ply  $(k)$ ,  $U_{I/II}^{(k0)} = \sum_{k=1}^N U_{I/II}^{(k)}$ .

The effect of the thermal residual stresses in the laminate, which are the result of cool-down after curing and the thermal expansion coefficient mismatch between different plies of the laminate, is included in the model by considering in eq. (13) the strain quantity that effectively gives strain energy of deformation in ply  $(k)$

$$\varepsilon^{(k)} = \varepsilon_{tot}^{(k)} - \alpha^{(k)} \cdot \Delta T \quad (15)$$

where  $\varepsilon_{tot}^{(k)}$  represents the *total* deformation of ply  $(k)$ , due to both mechanical and thermal loading.

The bending deformation is included by considering the linear variation of strain over the laminate thickness

$$\boldsymbol{\varepsilon}_{tot}^{(k)} = \boldsymbol{\varepsilon}_{tot}^{(0)} + \boldsymbol{z} \cdot \boldsymbol{k}_{tot} \quad (16)$$

The crack closure effect, that has to be accounted for in flexural deformation, is considered according to the sign of the transverse stress in ply ( $k$ )

$$\rho_{i+1}^{(k)} = \begin{cases} (i+2)\rho_0 & \text{if } \sigma_{22}^{(k)} > 0; \text{ crack is open (active)} \\ 0 & \text{if } \sigma_{22}^{(k)} \leq 0; \text{ crack is close (pasive)} \end{cases} \quad (17)$$

The experimentally observed *R-curve* behavior of the material resistance to crack multiplication [28] is also implemented in the present model. This phenomenon, that is specific to laminated composites, means an increasing resistance  $G_c$  of the material to crack growth with increasing damage level; it is attributed to the *fiber bridging effect* [29] during matrix crack multiplication process. According to this effect, the critical ERR in eq. (11) becomes  $G_c = G_c(\lambda)$ , where  $\lambda$  is the current damage level (crack density) of the ply. Moreover, for laminated composites there is also a thickness dependence of  $G_c$  [28], which is denoted in this model as the *in-situ R-curve* behavior. According to this, the critical ERR becomes  $G_c = G_c(\lambda, t)$  [24], [25]. In this work, only mode I critical SEER is accounted for the *in-situ R-curve* effect; for mode II, a constant value of  $G_c$  is used, due to lack of experimental data.

A first set of matrix cracking onset and multiplication experimental data, for a reference laminate having a cracking ply of thickness  $t = t_{ref}$ , is used to adjust the *R-curve* behavior of the reference laminate, which is modeled by the equation

$$G_{Ic}(\lambda, t = t_{ref}) = G_{Ic}^{ref} = G_{Ic,0}^{ref} + \tan(\beta_{\lambda}^{ref}) \cdot \lambda \quad (18)$$

where  $G_{Ic,0}^{ref}$  critical ERR for *onset* of damage (the advent of the first matrix cracking), and  $\beta_{\lambda}^{ref}$  is a parameter that describes the rate of damage growth, with  $0 \leq \beta_{\lambda}^{ref} < 90$ . Both  $G_{Ic,0}^{ref}$  and  $\beta_{\lambda}^{ref}$  parameters are identified by reduction of experimental data of the reference laminate.

A second set of matrix cracking onset and multiplication experimental data, for a laminate having a cracking ply of thickness  $t \neq t_{ref}$ , is considered next, in order to capture the *in-situ* (thickness) effect of the fracture toughness. Based on this, the *onset* fracture toughness is modeled as

$$G_{Ic,0}(t) = G_{Ic,0}^{ref} \cdot \left[ 1 + \left( \frac{t}{t_{ref}} - 1 \right) (\beta_0^t - 1) \right] \quad (19)$$

where  $\beta_0^t$  is an *in-situ onset* reduction data parameter, with  $1 \leq \beta_0^t \leq 2$ . Finally, the *in-situ R-curve* of the given material can be modeled based on the equation

$$G_{Ic}(\lambda, t) = G_{Ic,0}(t) + \left[ 1 + \left( \frac{t}{t_{ref}} - 1 \right) (\beta_{\lambda}^t - 1) \right] \cdot \tan(\beta_{\lambda}^{ref}) \cdot \lambda \quad (20)$$

where  $\beta_{\lambda}^t$  is an *in-situ growth* reduction data parameter, with  $1 \leq \beta_{\lambda}^t \leq 2$ .

The output of the progressive matrix cracking analytical model is:

- the reduced material properties of the laminate as a function of damage level in any ply ( $k$ ):  $C = C(\lambda^{(k)})$ ;
- the curve of damage multiplication under external loading for each ply ( $k$ ) of the laminate:  $\lambda^{(k)} = \lambda^{(k)}(\varepsilon)$ ;
- the curve of reduction in thermo-elastic properties of the laminated composite under external loading:  $C = C(\varepsilon)$ ;
- the nonlinear constitutive stress--strain behavior of the laminate material:  $\sigma = \sigma(\varepsilon)$ .

## VI. RESULTS AND CONCLUSIONS

The progressive damage matrix cracking experimental results presented in [18] are used here for the validation of the analytical model predictions. The material is a Glass/Epoxy *Fibredux 913G-E-5-30%* system, having the following thermo-elastic properties of individual plies:  $E_1 = 34 \text{ GPa}$ ,  $E_2 = 18 \text{ GPa}$ ,  $G_{12} = 7.9 \text{ GPa}$ ,  $\nu_{12} = 0.29$ ,  $\nu_{23} = 0.41$ ,  $\alpha_1 = 6.72 \times 10^{-6} \text{ }^\circ\text{C}$ ,  $\alpha_2 = 29.3 \times 10^{-6} \text{ }^\circ\text{C}$ . The ply thickness is  $t_k = 0.125 \text{ mm}$ , and the processing temperature (stress free temperature) is  $SFT = 125 \text{ }^\circ\text{C}$ . Two experimental data sets, namely [0/90/0] and [0/90<sub>2</sub>/0] are used for the calculation of the *in-situ R-curve* parameters, yielding the following values:

$$G_{Ic,0}^{ref} = 0.2 \text{ N/mm}, \quad \beta_{\lambda}^{ref} = 0 \text{ deg}, \quad \beta_0^t = 1, \quad \beta_{\lambda}^t = 1.$$

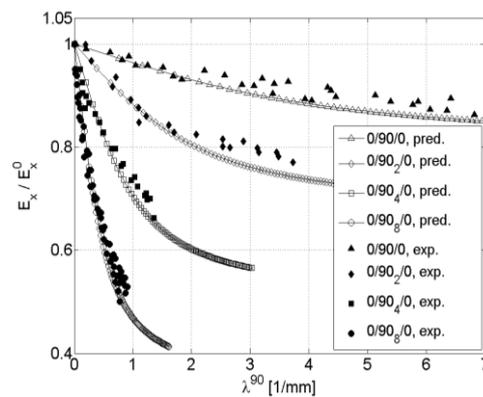


Fig. 1. Variation of normalized Young modulus with crack density; experimental data from [18].

A first set of experimental results is presented in [18] for the [0/90<sub>n</sub>/0] LSS under uniaxial extension  $\varepsilon_x$ . Reduced material properties of the laminate are presented in Fig. 1, 2, for the Young modulus and Poisson ratio, respectively. Good agreement between analytical and experimental results can be noticed in the two plots. However, it can also be noticed the model tendency of predicting a lower slope of the

analytical trend compared to the experimental trend, at high values of the crack density  $\lambda$ , especially for high thickness of the cracking plies. The matrix cracking evolution curves are presented in Fig. 3. Again, the least favorable match of the experimental data can be noticed for the thick cracking plies, at high crack densities.

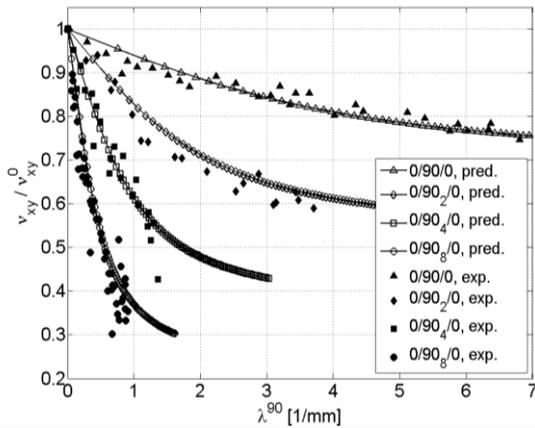


Fig. 2. Variation of normalized Poisson ratio with crack density; experimental data from [18].

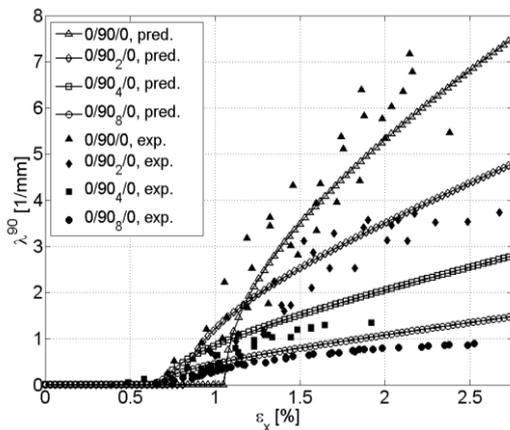


Fig. 3. Variation of crack density with applied strain; experimental data from [18].

Stress-strain curves featuring nonlinearities due to matrix cracking are shown in Fig. 4, where good agreement between analytical and experimental results is obtained. The least favorable correlation is obtained for the [0/90<sub>8</sub>/0] laminate featuring the thickest cracking ply, which is in connection with the least favorable correlation of damage evolution curve in Fig.3. This deteriorating predicting capability of the model at high thickness of the cracking ply could be due to the so called *transition thickness* effect, experimentally noticed for laminated composites: there is a certain thickness (i.e., the *transition thickness*) of the ply, over which the strain at matrix cracking onset remains constant.

Experimental data for matrix cracking under flexural loading is very rare in the literature. A set of such data is presented in [18], for the LSS [90<sub>2</sub>/0<sub>2</sub>/-45<sub>2</sub>/45<sub>2</sub>]<sub>S</sub> and [90<sub>2</sub>/-45<sub>2</sub>/45<sub>2</sub>]<sub>S</sub>. The curves of distributed bending moment-curvature (which are the equivalent of the stress-strain plots for the case of extension loading) are presented in Fig. 5, where good agreement between analytical predictions and experiments can be noticed.

Model prediction for combined extension-bending loading  $\epsilon_x - k_x$  is presented in Fig. 6. The LSS considered for this

case is [0/90<sub>2</sub>/0/90<sub>3</sub>/0]<sub>S</sub>. Without using the symmetry notation, the laminate is denoted as [0/90<sub>2</sub><sup>(1)</sup>/0/90<sub>3</sub><sup>(1)</sup>/0<sub>2</sub>/90<sub>3</sub><sup>(2)</sup>/0/90<sub>2</sub><sup>(2)</sup>/0]; the superscripts (1) denotes plies at the bottom of the LSS (which would be under compression if only bending deformation was applied), and the superscripts (2) denotes plies at the top of the LSS (which would be under tension if only bending deformation was applied). The applied loading is simultaneous extension  $\epsilon_x$  and bending  $k_x$ , under displacement control; the ratio of the two applied incremental deformations is  $k_x = 0.5 \cdot \epsilon_x$  for the present study.

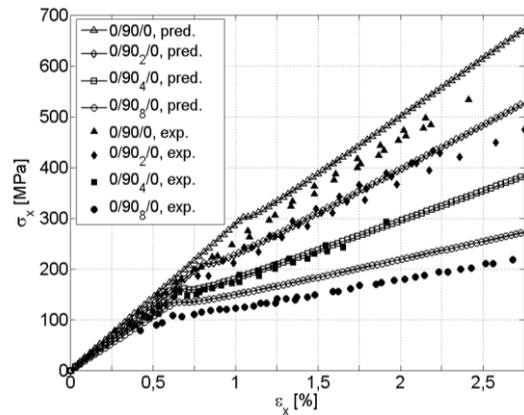


Fig. 4. Non-linear stress-strain curves due to matrix cracking; experimental data from [18].

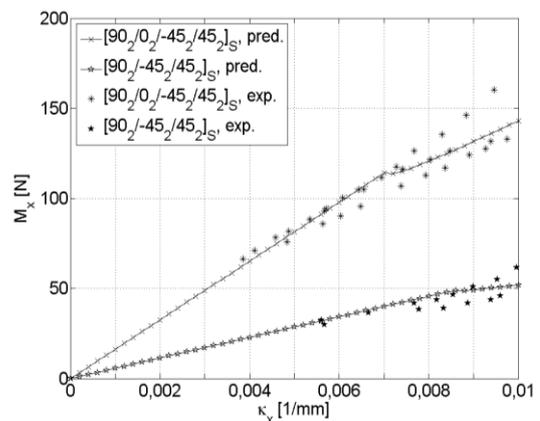


Fig. 5. Non-linear distributed bending moment vs. curvature due to matrix cracking; experimental data from [18].

It can be observed in Fig. 6 that the cracking process initiates almost simultaneous in the top 90<sub>2</sub><sup>(2)</sup> and 90<sub>3</sub><sup>(2)</sup> plies, due to combined effect of ply thickness and bending deformation. Regarding the damage process in the bottom 90<sub>2</sub><sup>(1)</sup> and 90<sub>3</sub><sup>(1)</sup> plies, it can be observed that there is matrix cracking in the 90<sub>3</sub><sup>(1)</sup> ply, and there is no matrix cracking in the 90<sub>2</sub><sup>(1)</sup> ply. The extensional strain exceeds the negative bending strain due to bending in both plies. However, there is more net strain in the 90<sub>3</sub><sup>(1)</sup> ply than in the 90<sub>2</sub><sup>(1)</sup> ply because of their position in the LSS, and the 90<sub>3</sub><sup>(1)</sup> is weaker because it is thicker (in-situ effect). This is the reason why matrix

cracking appears in the  $90_3^{(1)}$  ply and it does not appear in the  $90_2^{(1)}$  ply.

In conclusion, the present model for progressive matrix cracking of general laminates, under combined membrane-flexural deformation, accounts for multiple aspects of the problem: the effect of thermal residual stresses, the crack closure effect in bending, and the in-situ R-curve effect. Two experimental data sets of damage growth, for two laminate configurations of the same material system, are needed in order to fit the R-curve parameters; then, the model is able to predict damage onset and evolution for other LSS. There is no limitation on the configuration of the laminate or on the number of the cracking plies, as it is the case of the most models available in the literature, where only symmetric LSS are addressed, or only certain plies can undergo matrix cracking. The trends output by the analytical model, as well the comparison against experimental data, provides confidence in the predicting capabilities on the proposed model.

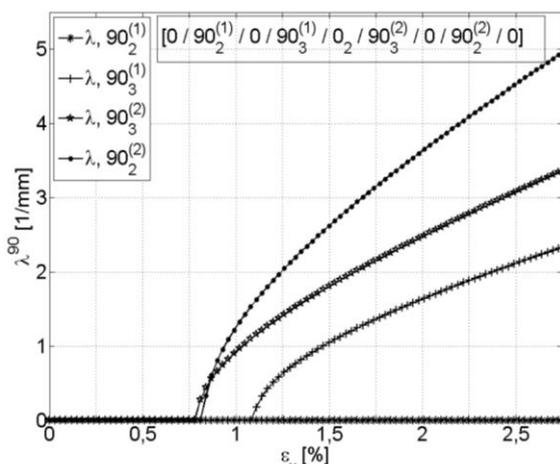


Fig. 6. Analytical variation of crack density for combined membrane-flexural loading.

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