INFLUENCE OF BENDING-EXTENSION COUPLING ON BUCKLING OF COMPOSITE COLUMNS

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Stability analysis of laminated composite columns with bending-extension coupling is presented. Previous work by other authors considered that the stability of the column is governed by a symmetric bifurcation and used a "reduced bending stiffness." A perturbation analysis is employed in this work to investigate the primary equilibrium path using linear, quadratic, and cubic approximations. Equilibrium and stability of the column is studied using the total potential energy approach. The results show that bending-extension coupling causes a nonstraight prebuckling equilibrium path. No critical states are detected along a cubic path, thus excluding the possibility of bifurcation into a half-wave mode or into a one-wave mode shape.

Fiber-reinforced polymer and metal-matrix composites have been used in aerospace and civil engineering applications [1] where high strength-to-weight ratios, corrosion resistance, or other features are required. Most structural applications use laminated composite members, with the laminated structure optimized for a particular application. As a result of different requirements on both sides of a laminate (thermal, abrasion, impact protection, etc.), it is not uncommon to have unsymmetrical laminates, which are modeled using bending-extension coupling stiffnesses. In the case of columns, unsymmetrical sections can be the result of unsymmetrical geometry as well as material properties (e.g., carbon fiber reinforcement of fiberglass beams, which are subsequently used also as columns). Compression members such as columns or panels are an essential part of most structural applications.

The design of composite columns and compression panels is usually governed by the buckling response. The equilibrium paths of a system having bifurcation-type bucking is shown in Figure 1, where λ is the load parameter and Q is an adequate displacement

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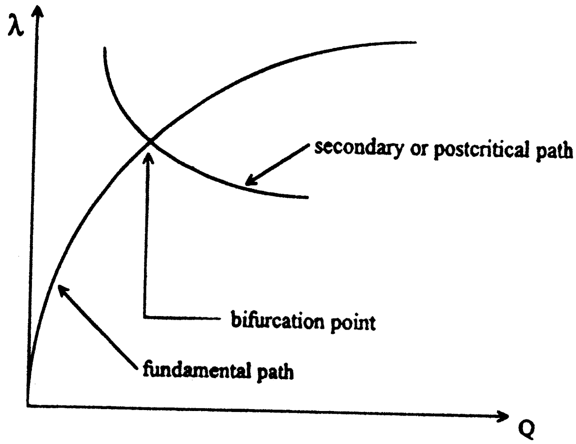

Figure 1. Schematic of the set of equilibrium states (path) of the column.

component. A path is formed by a set of equilibrium states; in complex problems such as the one discussed in this article, there may be more than one path. This is illustrated in Figure 1, where the fundamental (also called the prebuckling or primary) path emerges from the origin in the λ–Q space; and the secondary (also called postcritical or postbuckling) path crosses the fundamental path at a point. This point is usually called a bifurcation (or critical) point. Another form in which a system may display buckling is in a limit-point type of behavior. A detailed discussion on the basic concepts of buckling may be found in the books by Croll and Walker [2], Thompson and Hunt [3], and others.

By using a nonlinear finite-element analysis of imperfect plates, Barbero and Reddy [4] show that the behavior of unsymmetrical plates deviates significantly from the classical bifurcation and stable postbuckling behavior of symmetric plates. However, they did not investigate the existence of bifurcation points on the nonlinear paths. For symmetric laminates, the prebuckling response is trivial in the sense that no lateral deflections are present if there are no initial imperfections, and the postbuckling response is stable, without imperfection sensitivity.

Because composite materials can experience large strains with linear elastic behavior, the stability analysis reduces to the determination of the bifurcation point (buckling load). In his pioneering work, Whitney [5] utilized the equilibrium method [6] to perform bifurcation analysis of plates, and the results are well known in the literature [7]. A trivial fundamental path is assumed in the application of the equilibrium method to this problem. When the laminate is unsymmetrical, the prebuckling solution is not trivial because of the presence of the bending-extension coupling matrix $[B]$ for plates [7] or bending-extension coupling coefficient B for columns [8, 9].

Deflections and vibration frequencies of simply supported, unsymmetrical beams under axial load are presented by Vinson and Sierakowski [10] in terms of reduced axial and bending stiffness. The reduced axial and bending stiffness were found by uncoupling the axial and bending beam governing equations, both of which were considered to be linear. As pointed out by Sun and Chin [11], nonlinear effects must be included in the analysis of unsymmetrical composite laminates. In this case, the prebuckling path is not trivial because of bending-extension coupling. A nontrivial prebuckling path is in general nonlinear, the stability of which must be investigated to determine the critical points (limit or bifurcation). Information about the postcritical path is required to assess the sensitivity to imperfections. Therefore, the objective of this work is to investigate the influence of bending-extension coupling on the stability of unsymmetrical laminates. The simpler case of columns was chosen because it allows us to conduct a thorough stability analysis without resorting to numerical methods of approximation.

In the next section, a formulation in terms of the total potential energy of the column is presented, in which the constitutive equations reflect the coupling between bending and extensional effects. Then a discrete form of the energy is obtained using an approximation in terms of three generalized coordinates. Next the fundamental path of equilibrium is presented using perturbation theory. This allows the computation of the path as a linear, quadratic, or cubic curve in the load-displacement space. The formulation to obtain critical states is then discussed, and numerical results are presented in order to highlight the main features of the behavior of these columns. Finally, some conclusions are presented.

BASIC FORMULATION

The study of the stability of unsymmetrical columns is carried out in this article following the general stability theory [3, 12], in which use is made of the total potential energy of the system.

Total Potential Energy

We consider a column as in Figure 2, under axial load and made of a composite material. For the sake of simplicity, boundary conditions are assumed as simply supported. The total potential energy of the elastic system may be written as

$$V = \frac{1}{2}\int_{x=0}^{l} (N\epsilon + M\chi + 2\lambda u')\, dx \tag{1}$$

The stress resultants N, M are related to the strains ϵ, χ by means of the constitutive relations

$$\begin{aligned} N &= A\epsilon + B\chi \\ M &= B\epsilon + D\chi \end{aligned} \tag{2}$$

where A is the membrane stiffness, D is the bending stiffness, and B represents the bending-extension coupling. Notice that the coefficients A, B, and D are not obtained from a one-dimensional laminate analysis ($A \neq A_{11}, B \neq B_{11}, D \neq D_{11}$). A useful discussion

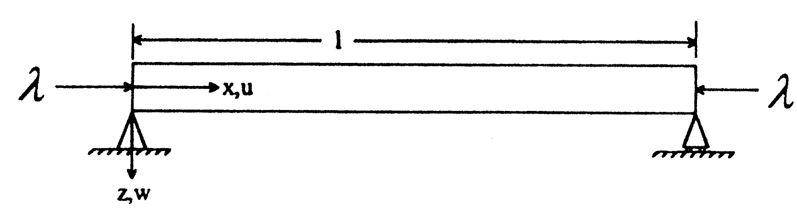

Figure 2. Geometry, boundary conditions, and load considered.

on this issue can be found in Whitney [13]. Expressions for the beam stiffness coefficients can be found in Tsai [14]. For the case of thin-walled beams, the coefficients can be found in Barbero et al. [9]. The nonlinear kinematic relations are assumed as

$$\begin{aligned} \epsilon &= u' + \tfrac{1}{2}(w')^2 \\ \chi &= -w'' \end{aligned} \tag{3}$$

with u and w the displacements shown in Figure 1, and $(\#)' = d(\#)/dx$. Substitution of Eqs. (2) and (3) in Eq. (1) leads to

$$\begin{aligned} V[u, w, P] = \tfrac{1}{2}\int_{x=0}^{l} &\left\{A\left[(u')^2 + (u')(w')^2 + \frac{(w')^4}{4}\right] + D(w'')^2 + 2\lambda(u')\right\} \\ &dx - \tfrac{1}{2}\int_{x=0}^{l} B[2(u')(w'') + (w'')(w')^2]\, dx \end{aligned} \tag{4}$$

The bending-extension coupling coefficient B is present only in the last term of Eq. (4), and multiplies quadratic and cubic terms in displacements. The functional is quartic in w and linear in the load term. To obtain the boundary conditions of the problem we invert the constitutive equations [Eq. (2)], leading to

$$\epsilon = \frac{1}{DA - B^2}(DN - BM) \tag{5}$$

$$-w'' = \chi = \frac{1}{DA - B^2}(-BN + AM) \tag{6}$$

The classical boundary conditions are

$$\begin{aligned} w(x = 0) &= 0 \\ w(x = l) &= 0 \end{aligned} \tag{7}$$

and

$$\begin{aligned} N(x = 0) &= -\lambda \\ N(x = l) &= -\lambda \end{aligned} \tag{8}$$

$$\begin{aligned} M(x = 0) &= 0 \\ M(x = l) &= 0 \end{aligned} \tag{9}$$

Substitution of Eqs. (8) and (9) into Eq. (6) leads to

$$
\begin{aligned}
w''(x = 0) &= -\frac{\lambda B}{DA - B^2} = -\alpha^2 \frac{B}{A} \\
w''(x = l) &= -\frac{\lambda B}{DA - B^2} = -\alpha^2 \frac{B}{A}
\end{aligned}
\tag{10}
$$

where

$$
\alpha^2 = \frac{\lambda A}{AD - B^2}
$$

Notice that Eq. (7) is the essential boundary conditions, and Eq. (10) is the natural boundary conditions of the problem. Because of bending extension coupling B, the curvature at the supports is not zero even though the applied moment is zero.

Approximate Solution

The equilibrium conditions of this problem are obtained from the first variation of V in Eq. (4) subject to the boundary conditions of Eqs. (7) and (10). The analytical solution of this problem in terms of the displacement components u and w is not available. Therefore, we have to employ an approximate solution.

To keep the problem simple, we have chosen to use a solution in terms of three parameters, Q_1, Q_2, and Q_3, which will be called the generalized coordinates of the problem. In this way, it is possible to carry out all the analysis analytically without any additional numerical approximation. The displacement field is written in terms of the generalized coordinates as

$$
\begin{aligned}
u' &= Q_1 + \phi(x) Q_2 \\
w &= [\phi(x) - 1] Q_3
\end{aligned}
\tag{11}
$$

where

$$
\phi(x) = \sin\left(\frac{\pi x}{2l}\right) + \cos\left(\frac{\pi x}{2l}\right) \tag{12}
$$

The mode shape $[\phi(x) - 1]$ in Eq. (11) is not identical to the mode shape of an Euler column made with a homogeneous material, but it satisfies the boundary conditions of Eqs. (7) and (10). In Eq. (11), Q_1 represents the classical end shortening, Q_2 the end shortening due to bending-extension coupling, and Q_3 the lateral deflection. The shape

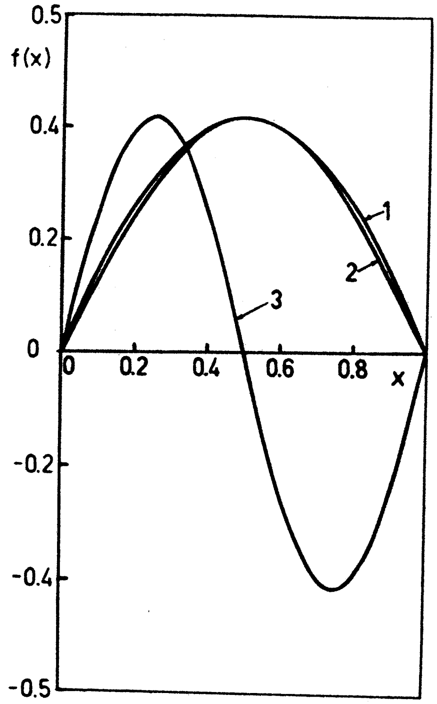

Figure 3. Displacement functions considered: 1, $f(x) = \sin(\pi x/2l) + \cos(\pi x/2l) - 1$; 2, $f(x) = \sin(\pi x/l)$; 3, $f(x) = \sin(2\pi x/l)$.

of Eq. (11) is very similar to $\sin(\pi x/l)$, as shown in Figure 3, and could be substituted by it with no serious consequences for the analysis.

Use of Eq. (11) in Eq. (4) and after integration leads to the total potential energy in terms of the generalized coordinates of the problem. This constitutes an expression with terms that are constant, linear, quadratic, cubic, and quartic in terms of the Q_i. A convenient way to write V is in the form

$$V[Q_1, Q_2, Q_3, \lambda] = \lambda \hat{V}_i' Q_i + \frac{1}{2!}\hat{V}_{ij} Q_i Q_j + \frac{1}{3!}\hat{V}_{ijk} Q_i Q_j Q_k + \frac{1}{4!}\hat{V}_{ijkl} Q_i Q_j Q_k Q_l \tag{13}$$

where $i = 1, 2, 3$; λ is the load. This form of writing V in Eq. (13) has been used by many authors in the context of the theory of elastic stability, for example, Croll and Walker [2]. The coefficients in the expansion of Eq. (13) are given by

$$\begin{aligned} \hat{V}_1' &= l \\ \hat{V}_2' &= \frac{4l}{\pi} \\ \hat{V}_{11} &= Al \\ \hat{V}_{12} &= \frac{4Al}{\pi} \end{aligned} \tag{14}$$

$$\hat{V}_{13} = \frac{\pi B}{l}$$

$$\hat{V}_{22} = \frac{(2 + \pi)Al}{\pi}$$

$$\hat{V}_{23} = \frac{(2 + \pi)\pi B}{4l}$$

$$\hat{V}_{33} = \frac{(2 + \pi)\pi^3 D}{16l^3}$$

$$\hat{V}_{133} = \frac{(\pi - 2)\pi A}{4l} \qquad \text{(14 Cont.)}$$

$$\hat{V}_{233} = \frac{\pi A}{3l}$$

$$\hat{V}_{333} = \frac{\pi^3 B}{4l^3}$$

$$\hat{V}_{3333} = 6\pi^3\left(\frac{3}{8}\pi - 1\right)\frac{A}{(2l)^3}$$

The $\hat{V}$ coefficients are symmetric, and those that are not listed are zero.

Notice that, in the present problem, the fundamental (prebuckling) path is nonlinear (because of bending-extension coupling); thus the buckling loads should be calculated from a nonlinear path, leading to a nonlinear eigenvalue problem. Finally, the postcritical paths are nonlinear. This is clearly not a simple column buckling problem, and to solve it, it is convenient to use a general theory for the elastic stability of discrete systems. Such a theory is fully developed by Thompson and Hunt [3] and Flores and Godoy [15], among others.

FUNDAMENTAL PATH

Perturbation Equations

The equilibrium condition may be written using the principle of stationary total potential energy in the form

$$\frac{\partial V}{\partial Q_i} = V_i = 0 \qquad (15)$$

In terms of the equilibrium equation, Eq. (13), this leads to

$$V_i = \hat{V}_i'\lambda + \hat{V}_{ij}Q_j + \tfrac{1}{2}\hat{V}_{ijk}Q_jQ_k + \tfrac{1}{6}\hat{V}_{ijkl}Q_jQ_kQ_l = 0 \qquad (16)$$

This is a cubic problem in terms of Q_i, with linear load parameter λ. Solution of the fundamental path can be done using perturbation techniques [3, 15].

Because of bending-extension coupling, the fundamental path has out-of-plane displacements Q_3 from the beginning of the loading process. One can adopt Q_3 as perturbation parameter ($s = Q_3$) and write the solution of the fundamental path in the form

$$
\begin{aligned}
Q_1 &= \dot{Q}_1 s + \frac{1}{2!}\ddot{Q}_1 s^2 + \frac{1}{3!}\dddot{Q}_1 s^3 + \dots \\
Q_2 &= \dot{Q}_2 s + \frac{1}{2!}\ddot{Q}_2 s^2 + \frac{1}{3!}\dddot{Q}_2 s^3 + \dots \\
Q_3 &= s \\
\lambda &= \dot{\lambda} s + \frac{1}{2!}\ddot{\lambda} s^2 + \frac{1}{3!}\dddot{\lambda} s^3 + \dots
\end{aligned}
\tag{17}
$$

where $(\dot{\#}) = d(\#)/ds = d(\#)/dQ_3$. Notice that we assume that for $Q_3 = 0$, the values of $Q_1 = Q_2 = \lambda = 0$ (the perturbation starts at the origin in the $\lambda - Q_i$ space). Using regular perturbation of Eq. (16) leads to the first-order perturbation equations:

$$
V_{ij}\dot{Q}_j + V'_i\dot{\lambda} = 0 \tag{18}
$$

The second-order perturbation equations are

$$
V_{ij}\ddot{Q}_j + V'_i\ddot{\lambda} = -2V'_{ij}\dot{\lambda}\dot{Q}_j - V_{ijk}\dot{Q}_j\dot{Q}_k - V''_i(\dot{\lambda})^2 \tag{19}
$$

and the third-order perturbation equations are

$$
\begin{aligned}
V_{ij}\dddot{Q}_j + V'_i\dddot{\lambda} = &-3V_{ijk}\dot{Q}_j\ddot{Q}_k - V''_i\dot{\lambda}\ddot{\lambda} - V'_{ij}(\dot{\lambda}\ddot{Q}_j + \ddot{\lambda}\dot{Q}_j) \\
&- V'_{ijk}\dot{\lambda}\dot{Q}_j\dot{Q}_k - V_{ijkl}\dot{Q}_j\dot{Q}_k\dot{Q}_l
\end{aligned}
\tag{20}
$$

where the following notation for derivatives of V is used:

$$
\begin{aligned}
\frac{\partial^2(V)}{\partial Q_i\,\partial Q_j} &= V_{ij} \\
\frac{\partial^3(V)}{\partial Q_i\,\partial Q_j\,\partial Q_k} &= V_{ijk} \\
\frac{\partial^4(V)}{\partial Q_i\,\partial Q_j\,\partial Q_k\,\partial Q_l} &= V_{ijkl} \\
\frac{\partial V}{\partial \lambda} &= V' \\
\frac{\partial^2(V)}{\partial \lambda^2} &= V''
\end{aligned}
$$

Evaluating Eqs. (18), (19), and (20) at $s = 0$, we obtain the following evaluated perturbation equations of order 1, 2, and 3:

$$\hat{V}_{ij}\dot{Q}_j + \hat{V}'_i\dot{\lambda} = 0 \tag{21}$$

$$\hat{V}_{ij}\ddot{Q}_j + \hat{V}'_i\ddot{\lambda} = -\hat{V}_{ijk}\dot{Q}_j\dot{Q}_k \tag{22}$$

$$\hat{V}_{ij}\dddot{Q}_j + \hat{V}'_i\dddot{\lambda} = -3\hat{V}_{ijk}\dot{Q}_j\ddot{Q}_k - \hat{V}_{ijkl}\dot{Q}_j\dot{Q}_k\dot{Q}_l \tag{23}$$

Solution of these equations leads to the unknown parameters required for the approximate perturbation solution of Eq. (17).

First-Order Perturbation Solution

From Eq. (21), it is possible to obtain an explicit solution for $\dot{Q}_j$ and $\dot{\lambda}$ as

$$\begin{aligned}
\dot{\lambda} &= \frac{\pi^2}{16}(2+\pi)\frac{AD - B^2}{Bl^2} \\
\dot{Q}_1 &= -\frac{\pi^2}{16}\frac{(\pi+2)}{l^2}\frac{AD - B^2}{AB} \\
\dot{Q}_2 &= -\frac{\pi^2}{4}\frac{B}{Al^2}
\end{aligned} \tag{24}$$

The perturbation parameter, s, was chosen as one of the displacement components in the derivation of Eqs. (17)–(24). An alternative would be to choose s on the load parameter λ, in which case $s = \lambda$ and

$$\begin{aligned}
\dot{Q}_1 &= \left(\frac{dQ_1}{d\lambda}\right)_{\lambda=0} = l \\
\dot{Q}_2 &= \left(\frac{dQ_2}{d\lambda}\right)_{\lambda=0} = \frac{4}{2+\pi}\frac{B^2 l}{(AD - B^2)} \\
\dot{Q}_3 &= \left(\frac{dQ_3}{d\lambda}\right)_{\lambda=0} = -\frac{16}{\pi^2(2+\pi)}\frac{ABl^3}{(AD - B^2)}
\end{aligned} \tag{25}$$

However, such choice of perturbation parameter has the disadvantage that it cannot represent a limit point behavior when higher-order perturbation systems are solved. Since we are investigating whether the column shows a bifurcation or a limit-point behavior, the capability to model both types of response must be retained.

It is interesting to observe that in Eq. (24), when $B^2 = AD$, then $\dot{\lambda} = d\lambda/dQ_3 = 0$, which indicates a zero slope in the load-deflection diagram at the origin ($\lambda = 0$, $Q_i = 0$). Therefore, $B^2 = AD$ is the case of lowest stiffness in the fundamental path.

Second-Order Perturbation Solution

From Eq. (22) one obtains

$$\begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}'_1 \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}'_2 \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}'_3 \end{bmatrix} \begin{Bmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \\ \ddot{\lambda} \end{Bmatrix} = \begin{Bmatrix} -\hat{V}_{133} \\ -\hat{V}_{233} \\ -2(\hat{V}_{133}\dot{Q}_1 + \hat{V}_{233}\dot{Q}_2) - \hat{V}_{3333} \end{Bmatrix} \tag{26}$$

Then Eq. (26) is solved for $\ddot{Q}_1$, $\ddot{Q}_2$, and $\ddot{\lambda}$, leading to

$$\ddot{Q}_1 = -3.1716\left(\frac{AB - B^2}{ABl^2}\right)$$

$$\ddot{Q}_2 = 6.0975\left(\frac{1}{l^2}\right)$$

$$\ddot{\lambda} = -1.81\left(\frac{AD - B^2}{l^2}\right)\left(\frac{A}{B^2}\right)$$

Third-Order Perturbation Solution

In a similar way, Eq. (23) can be written as

$$\begin{bmatrix} \hat{V}_{11} & \hat{V}_{12} & \hat{V}'_1 \\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}'_2 \\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}'_3 \end{bmatrix} \begin{Bmatrix} \dddot{Q}_1 \\ \dddot{Q}_2 \\ \dddot{\lambda} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -3(\hat{V}_{133}\ddot{Q}_1 + \hat{V}_{233}\ddot{Q}_2) - \hat{V}_{3333} \end{Bmatrix} \tag{27}$$

and this system is solved for $\dddot{Q}_1$, $\dddot{Q}_2$, and $\dddot{\lambda}$, leading to

$$\dddot{Q}_1 = -1.549\left(\frac{A}{B^3}\right)\left(\frac{AD - B^2}{l^2}\right)$$

$$\dddot{Q}_2 = 0$$

$$\dddot{\lambda} = 1.549\left(\frac{A^2}{B^3}\right)\left(\frac{AD - B^2}{l^2}\right)$$

CRITICAL STATES IN THE NONLINEAR FUNDAMENTAL PATH

Critical states (either limit or bifurcation points) are identified along the fundamental path by the condition [15]

$$(V_{ij}X_j)^c = 0 \tag{28}$$

where X_j is the eigenvector associated to the eigenvalue λ. From the derivation of Eq. (16), one gets

$$V_{ij} = \hat{V}_{ij} + \hat{V}_{ijk}Q_k + \tfrac{1}{2}\hat{V}_{ijkl}Q_kQ_l \tag{29}$$

Substitution of the cubic fundamental path, Eq. (17), into Eq. (29) leads to

$$\begin{aligned} V_{ij} = \hat{V}_{ij} &+ \hat{V}_{ijk}(\dot{Q}_k s + \tfrac{1}{2}\ddot{Q}_k s^2 + \tfrac{1}{6}\dddot{Q}_k s^3) \\ &+\tfrac{1}{2}\hat{V}_{ijkl}(\dot{Q}_k s + \tfrac{1}{2}\ddot{Q}_k s^2 + \tfrac{1}{6}\dddot{Q}_k s^3) \\ &\times (\dot{Q}_l s + \tfrac{1}{2}\ddot{Q}_l s^2 + \tfrac{1}{6}\dddot{Q}_l s^3) \end{aligned}$$

or else

$$\begin{aligned} V_{ij} = \hat{V}_{ij} &+ s(\hat{V}_{ijk}\dot{Q}_k) + \tfrac{1}{2}s^2(\hat{V}_{ijk}\ddot{Q}_k + \hat{V}_{ijkl}\dot{Q}_k\dot{Q}_l) \\ &+ s^3(\tfrac{1}{2}\hat{V}_{ijkl}\ddot{Q}_k\dot{Q}_l + \tfrac{1}{6}\hat{V}_{ijk}\dddot{Q}_k) \\ &+ s^4(\tfrac{1}{8}\hat{V}_{ijkl}\ddot{Q}_k\ddot{Q}_l + \tfrac{1}{6}\hat{V}_{ijkl}\dot{Q}_k\dddot{Q}_l) \\ &+ s^5(\tfrac{1}{12}\hat{V}_{ijkl}\ddot{Q}_k\dddot{Q}_l) + s^6(\tfrac{1}{12}\hat{V}_{ijkl}\dddot{Q}_k\dddot{Q}_l) \end{aligned}$$

Since only $\hat{V}_{3333} \neq 0$, and $\dot{Q}_3 = 1$, $\ddot{Q}_3 = \dddot{Q}_3 = 0$, then V_{ij} reduces to

$$V_{ij} = \hat{V}_{ij} + s(\hat{V}_{ijk}\dot{Q}_k) + \tfrac{1}{2}s^2(\hat{V}_{ijk}\ddot{Q}_k + \hat{V}_{ijkl}\dot{Q}_k\dot{Q}_l) + \tfrac{1}{6}s^3(\hat{V}_{ijk}\dddot{Q}_k) \tag{30}$$

But retaining all terms in Eq. (30) would not be consistent with the original energy functional V, which was only quartic in Q_i. This means that V_{ij} can only be quadratic in displacements. A consistent second variation of the energy, as represented by V_{ij}, is

$$V_{ij} = \hat{V}_{ij} + s(\hat{V}_{ijk}\dot{Q}_k) + \frac{s^2}{2}(\hat{V}_{ijk}\ddot{Q}_k) \tag{31}$$

and this is adopted in the rest of the computations.

The V_{ij} are given in explicit form as

$$\begin{aligned} V_{11} &= \hat{V}_{11} \\ V_{12} &= \hat{V}_{12} \\ V_{13} &= \hat{V}_{13} + s\hat{V}_{133} \\ V_{22} &= \hat{V}_{22} \\ V_{23} &= \hat{V}_{23} + s\hat{V}_{233} \end{aligned} \tag{32}$$

$$V_{33} = \hat{V}_{33} + s(\hat{V}_{331}\dot{Q}_1 + \hat{V}_{233}\dot{Q}_2 + \hat{V}_{333}) + \frac{s^2}{2}(\hat{V}_{331}\ddot{Q}_1 + \hat{V}_{233}\ddot{Q}_2 + \hat{V}_{3333}) + \frac{s^3}{6}(\hat{V}_{331}\dddot{Q}_1)$$

The determinant of V_{ij} can now be computed as

$$\begin{aligned}
\det(V_{ij}) = {} & s^3(\tfrac{1}{6}\hat{V}_{11}\hat{V}_{22}\hat{V}_{133}\dddot{Q}_1 - \tfrac{1}{6}\hat{V}_{12}\hat{V}_{133}\dddot{Q}_1) \\
& + s^2(\hat{V}_{11}\hat{V}_{233}^2 - \tfrac{1}{2}\hat{V}_{12}^2\hat{V}_{3333} - \hat{V}_{133}^2\hat{V}_{22} - \tfrac{1}{2}\hat{V}_{12}^2\hat{V}_{133}\ddot{Q}_1 \\
& \quad - \tfrac{1}{2}\hat{V}_{12}^2\hat{V}_{233}\ddot{Q}_2 + \tfrac{1}{2}\hat{V}_{11}\hat{V}_{22}\hat{V}_{133}\ddot{Q}_1 + \tfrac{1}{2}\hat{V}_{11}\hat{V}_{22}\hat{V}_{3333} \\
& \quad + 2\hat{V}_{12}\hat{V}_{133}\hat{V}_{233} + \tfrac{1}{2}\hat{V}_{11}\hat{V}_{22}\hat{V}_{233}\ddot{Q}_2) \\
& + s(\hat{V}_{11}\hat{V}_{22}\hat{V}_{133}\dot{Q}_1 + \hat{V}_{11}\hat{V}_{22}\hat{V}_{233}\dot{Q}_2 - \hat{V}_{12}^2\hat{V}_{333} \\
& \quad - 2\hat{V}_{13}\hat{V}_{22}\hat{V}_{133} - \hat{V}_{12}^2\hat{V}_{233}\dot{Q}_2 + \hat{V}_{11}\hat{V}_{22}\hat{V}_{333} \\
& \quad - \hat{V}_{12}^2\hat{V}_{133}\dot{Q}_1 + 2\hat{V}_{12}\hat{V}_{13}\hat{V}_{233} + 2\hat{V}_{12}\hat{V}_{133}\hat{V}_{23} - \hat{V}_{11}\hat{V}_{23}\hat{V}_{233}) \\
& + (2\hat{V}_{12}\hat{V}_{13}\hat{V}_{23} + \hat{V}_{11}\hat{V}_{22}\hat{V}_{33} - \hat{V}_{11}\hat{V}_{23}^2 - \hat{V}_{12}^2\hat{V}_{33} - \hat{V}_{22}\hat{V}_{13}^2) \qquad (33)
\end{aligned}$$

It is interesting to note that many terms cancel, and the determinant is only cubic in s. Three values of critical displacements are determined from the determinant.

If, instead of using the cubic fundamental path, we employ only up to quadratic terms in Eq. (17), then the last term in Eq. (30) is not present. The determinant in this case is quadratic.

A further simplification can be achieved by including only linear terms in Eq. (17), which leads to the consistent approximation

$$V_{ij} = \hat{V}_{ij} + s(\hat{V}_{ijk}\dot{Q}_k)$$

In the cubic and quadratic approximations of the fundamental path, the eigenvalue problem is solved in terms of the critical value of s, which leads to the displacement Q_3^c. This value can be substituted in Eq. (17) to obtain the associated critical load λ_c, and the rest of the displacement components.

The eigenvector X_j^c results from Eq. (28); however, V_{ij}^c is singular and a component of X_j^c has to be assumed as a known value to compute the other components. In this work we have chosen $X_3^c = 1$, and X_2^c, X_1^c are computed from Eq. (28).

NUMERICAL RESULTS

Initial Remarks

For the case of beams and columns, it is worth noting that the expressions for axial, bending, and bending-extension coupling stiffness given by Vinson and Sierakowski [10] and Barbero [8] are only approximations because of the use of a plane stress assumption through the thickness of the beam. Exact stiffness coefficients, based on the condition of vanishing transverse stress resultant, can be found in Tsai [14] for rectangular beams and in Barbero et al. [9] for thin-walled beams.

Before presenting our own results, it is important to consider previous work in this field. For the present column, the accepted buckling load is given by Vinson and Sierakowski [10] as

$$\lambda_c = \left(\frac{\pi}{L}\right)^2 \left(\frac{AD - B^2}{A}\right) n^2 \qquad n = 1, 2, \ldots \tag{34}$$

where n is the number of half-waves considered in the out-of-plane displacement

$$w = Q \sin \frac{n\pi x}{L}$$

where Q is the amplitude, equivalent to our Q_3 in this article. Clearly, the lowest buckling load is given by $n = 1$, in which case

$$\lambda_c = \left(\frac{\pi}{L}\right)^2 \left(\frac{AD - B^2}{A}\right) \tag{35}$$

In Eq. (34), the reduced bending stiffness $D - B^2/A$ is equal to zero for a value of the bending-extension coupling coefficient $B^2 = AD$, which is an upper bound for the value of bending-extension coupling on a column. Furthermore, if we limit the type of columns to laminated rectangular sections, a new upper bound for the value of the bending-extension coupling can be found. It is well known [7] that the maximum coupling occurs for a laminate with two layers ($N = 2$). If the value of the orthotropic properties in one of the two layers is much larger than in the other layer, the stiffness coefficients can be approximated from classical lamination theory as

$$A \simeq btE \qquad B \simeq b\frac{t^2}{2}E \qquad D \simeq b\frac{t^3}{3}E \tag{36}$$

where E is the modulus of elasticity of the stiffer layer along the axis of the column, t is the thickness of one layer, and b is the width of the column. These expressions can be interpreted as having neglected the contribution of the less stiff layer. For the two-layer, rectangular column, the upper bound for the bending-extension coupling is $B^2 = \frac{3}{4}AD$. For the limit case in which only the stiffer layer is considered, it is possible to write an expression for the slenderness ratio (l/r) of the column in terms of the fixed values of A and D as

$$\gamma = \frac{l^2}{r^2} = l^2 \frac{A}{D} \tag{37}$$

where l is the length, r is the radius of gyration of the one-layer column, and γ is the square of the slenderness ratio.

Results from the Present Model

Numerical results are presented for values of the bending-extension coupling between the minimum value (zero for symmetric laminate) and the maximum value of the bending-extension coupling B^*, given by

$$B^* = \sqrt{\frac{3}{4} AD}$$

The columns analyzed correspond to an E-glass epoxy with $E = 38.586$ GPa, with rectangular section 10×3 mm^2, leading to $A = 1.15758$ MN, and $D = 3.47274$ Nm^2.

The accepted value of buckling load [Eq. (35)] in the first mode with a half-sine wave, for $B = 0.75B^*$, is given by

$$\frac{\lambda_c}{\lambda_{euler}} = 0.578$$

where λ_{euler} is the bifurcation load of the column with $B = 0$. A cubic perturbation analysis of the same case is shown in Figure 4, with load as a function of the out-of-plane displacement Q_3. The path shows increasing displacements with increasing load parameter, and there are no signs of limit points along it. Three roots are found for the determinant of V_{ij}, and they give two imaginary values and another one at infinity (a value of Q_3 of the order of 10^7).

The results show that there is no critical state in the form of a bifurcation with $n = 1$ [or in the form of Eq. (11) in this work]. The reason for that is simple: The

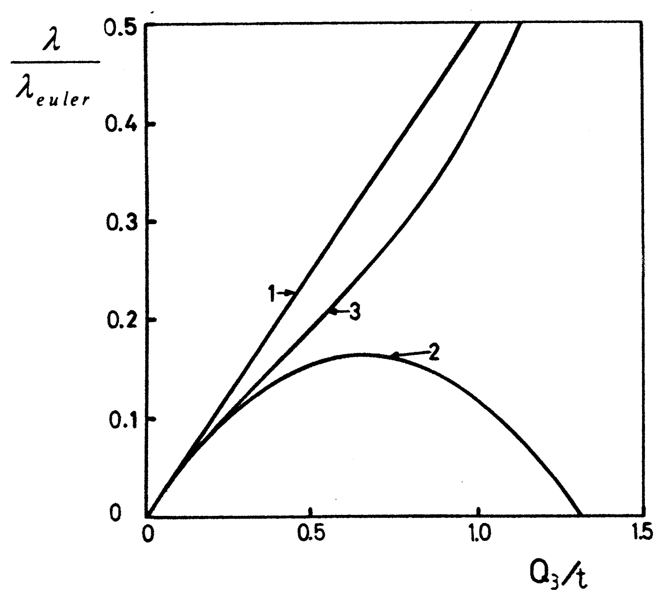

Figure 4. Load-deflection path for $B = 0.75B^*$: 1, linear; 2, quadratic; 3, cubic.

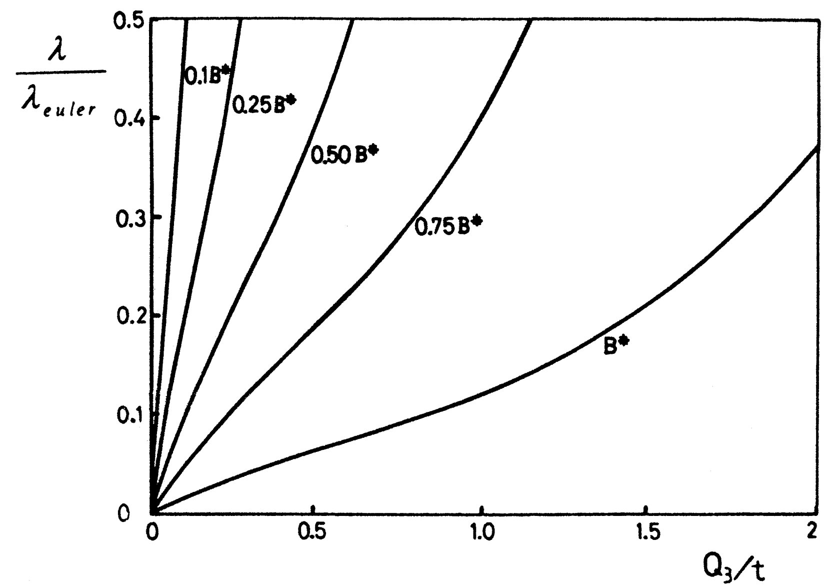

Figure 5. Cubic load-deflection path for various values of B.

fundamental path has a shape with one-half sine wave, and cannot bifurcate into the same shape.

What are the consequences of considering less accurate expressions for the fundamental path? To answer this, we retain up to quadratic terms in the fundamental path, and the results show the limit-point behavior of Figure 4. An even less accurate model with only linear terms leads to a linear fundamental path with a critical state that captures the limit point in the quadratic path. If the fundamental path is approximated with a trivial approximation ($Q_3 = 0$), then the apparent critical state is that given by Eq. (34). This shows the dangers with establishing erroneous approximations in the nonlinear fundamental path. In the exact path, which is closely followed by the cubic perturbation analysis, the out-of-plane displacements grow with the load and lead to a nonlinear stiffening behavior. Bifurcation points into a half-wave mode cannot be found along this path, and are not found in the linear and quadratic approximations. Neglecting vital cubic terms yields a nonlinear quadratic path with a limit point, which is not an acceptable approximation.

Next it is important to investigate the influence of the bending-extension coupling coefficient B on the solution. This is shown in Figure 5, for values of B between $0.1B^*$ and B^* as limiting cases. For low values of B, $B \to 0$ and the fundamental path tends to satisfy the condition $Q_3 = 0$, as in symmetric laminates. Notice that for a symmetric laminate there should be a bifurcation point into a half-wave mode. As the value of B increases, the fundamental path exhibits a cubic shape with initially decreasing stiffness, followed by a stiffening behavior.

Having discarded a bifurcation in the $n = 1$ mode, we investigate possible bifurcations into higher modes. For that, we consider the out-of-plane displacement represented by

$$w = Q_3 \sin \frac{\pi x}{L} + Q_4 \sin \frac{2\pi x}{L}$$

and investigate possible bifurcations into a mode with $n = 2$. The analysis carried out showed that there are no bifurcations into such modes.

CONCLUSIONS

The main purpose of this article is to investigate the influence of the coupling between bending and extensional stresses and strains, in a column made with a composite material, on the prebuckling and buckling states. A simple model of the column has been developed as an approximation to the physical problem, and the general theory of elastic stability of discrete systems has been used to investigate the buckling process.

From the results obtained, it is possible to conclude that the effect of bending-extension coupling is to induce a nonlinear fundamental path, even for initially perfect columns. There is an initial effect of unstiffening (in the sense that the stiffness is reduced from an initial value); and following an inflection point in the load-deflection curve, there is a stiffening behavior. As the bending-extension coupling coefficient is increased, the nonlinearity of the fundamental path is stronger. For those cases, a linear approximation to the fundamental path leads to poor results, which can only be accepted for small values of bending-extension coupling.

The fundamental path in Figures 4 and 5 is a cubic curve and has the same trend as the results for unsymmetrical plates obtained by Barbero and Reddy [4] using a nonlinear finite-element analysis.

The cubic equilibrium path shown here and the false linear bifurcation buckling that can be computed have been shown to occur in several problems in elastic stability. For example, in the nonlinear problem of an axially compressed cylindrical shell with diametrically opposed cutouts studied in Brush and Almroth [16], a linear bifurcation analysis underestimates collapse by more than a factor of 2. Also, the concern of the authors about the existence of bifurcation behavior in columns with bending-extension coupling has been shared by other studies on plates. According to Leissa [17], "some researchers have questioned whether, because of the bending-stretching coupling, meaningful bifurcation buckling problems can exist for an unsymmetrical laminate. . . . In this case an eigenvalue problem of bifurcation buckling would not arise but, rather an equilibrium problem similar in nature to those arising due to eccentric loadings or geometric imperfections."

The postcritical states investigated are all stable. Within the range of the perturbation expansion, all the postcritical paths are "rising": There are postbuckling equilibrium states for loads higher than the critical load. However, the postcritical path has a high curvature, so that an increase in the load can be obtained only with large changes in postcritical displacements. Considering the critical load level (bifurcation load) as the maximum load that the system can attain seems to be a reasonably safe engineering estimate.

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