Buckling mode interaction in composite plate assemblies

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The analysis of buckling mode interaction of fiber-reinforced composite columns, modeled as plate assemblies, is presented. The main assumptions are linear elasticity; a linear fundamental equilibrium path; the existence of critical states that are coincident or near coincident; and a coupled path rising from a quadratic combination of modal displacements due to interaction. The formulation adopted is known as the W-formulation, in which the energy is written in terms of a sliding set of incremental coordinates, measured with respect to the fundamental path. The energy is then expressed with respect to a reduced modal coordinate basis, and the coupled solution arising from interaction is computed. An example of a pultruded composite I-column subjected to axial compression illustrates the procedure.

INTRODUCTION

Fiber reinforced composite structural shapes are constructed using thin-walled cross sections to take advantage of the high strength of the material, and to compensate for its low modulus. In typical flange-web assemblies, the flanges may buckle locally, and the column may buckle globally, at loads that are close or even coincident for column lengths commonly used in practice. Experimental evidence (Barbero and Tomblin 1994) shows buckling loads lower than the global or local loads predicted from isolated mode analysis. Since composite materials remain elastic even for large strains, it was speculated by Barbero and Tomblin (1994) that the reduction in buckling strength may be caused by buckling mode interaction.

When two or more modes of buckling correspond to loads that are close or coincident, interaction between the modes may lead to post-buckling behavior quite different from the post-buckling behavior of the participating modes. The general theory of elastic stability for discrete systems (Thompson and Hunt 1973)was chosen in this work as the framework for the study of post-buckling behavior with the aid of finite element discretization. In this work, perturbation techniques are used to classify the nature of the bifurcation points and to follow the initial post-buckling path. Furthermore, interaction between modes emerging from bifurcation points, two of which are close or coincident, is studied to identify tertiary paths emerging from the secondary paths. The tertiary paths may be quite different from the secondary paths identified by an isolated mode analysis, thus revealing a different kind of imperfection sensitivity of the structure.

Mode interaction in discrete systems was studied by Chilver (1967), Supple (1967), Chilver and Johns (1971), Thompson and Supple (1973), Swanson and Croll (1975), Reis (1977), Reis and Roorda (1979), Maaskant (1989). Analytical methods were used by Koiter and Pignataro (1976) for stiffened panels and by Kolakowski (1993) to study trapezoidal columns. From the literature, it is possible to identified numerical problems that may arise while using continuation methods in mode-coupling problems.

The finite strip method was used by Mollmann and Goltermann (1989), while finite elements were used by Casciaro *et al* (1992). The finite element method was chosen in our work because of its superior versatility to model complex boundary conditions and because it automatically accounts for the problem of wave modulation (Koiter 1967).

There are several alternatives to carry out mode-coupling post-buckling analysis in composite columns with symmetric cross section. First, it is possible to combine a local and a global mode, then compute first order fields. This was done previously by Barbero *et al* (1993), who showed that this analysis is not sufficient to detect buckling mode interaction. Second, it is possible to combine a local and a global mode, and to compute first and second order fields. This is the methodology used in this paper. Another possibility is to

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combine three modes and to compute first order fields, as shown by Godoy *et al* (1994) along with a simplified analytical model.

In this paper, a general formulation is presented for the finite element analysis of plate assemblies, including buckling and mode interaction. The formulation is written in terms of the usual matrices used in finite element analysis, and it represents an extension of previous work by the authors for non-interacting (isolated) modes. Since we are primarily interested in applications to fiber reinforced composite columns, a shear deformable finite element is used to model the structure as an assembly of plate elements. The finite element implementation is capable of dealing with cross-sections of arbitrary shape, as well as symmetric cross sections. Numerical examples are presented for the latter case since it is considered a critical test case.

FORMULATION

The total potential energy V of a discrete system can be expressed as a Taylor expansion with respect to a set of generalized coordinates Q_i (usually nodal displacements) as (Raftoyiannis 1994)

$$V = \hat{V}_0+\hat{V}_iQ_i+\frac{1}{2}\hat{V}_{ij}Q_iQ_j +\frac{1}{6}\hat{V}_{ijk}Q_iQ_jQ_k+\frac{1}{24}\hat{V}_{ijkl}Q_iQ_jQ_kQ_l \quad (1)$$

where i, j, k, l = 1,..., N, N being the number of degrees of freedom of the system. The discrete system may result from a finite element discretization of a continuous system.

After finding the initial response, the nodal displacements can be written as

$$Q_i=\Lambda Q_i^F+q_i \quad (2)$$

where q_i are the incremental displacements, and Q_i^F is the response in the fundamental path for unit value of the load parameter. Introduction of Eq (2) into Eq (1) results in the W-energy that, after neglecting terms independent of q_i and terms with non-linear contribution from Q_i^F, can be written as

$$W=\hat{W}_iq_i+\frac{1}{2}\hat{W}_{ij}q_iq_j +\frac{1}{6}\hat{W}_{ijk}q_iq_jq_k+\frac{1}{24}\hat{W}_{ijkl}q_iq_jq_kq_l \quad (3)$$

where

$$\hat{W}_i=\hat{V}_i$$
$$\hat{W}_{ij}=\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F$$
$$\hat{W}_{ijk}=\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F$$
$$\hat{W}_{ijkl}=\hat{V}_{ijkl}$$

The condition of critical stability is

$$[\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F]\ x_j=0 \quad (4)$$

where x_j are the components of the eigenvector.

The solution of Eq (4) is a set of N eigenvalues, which we will denote as Λ^n (the superscript identifying the mode number), andeigenvectors $\mathbf{x}^n$, for n = 1,..., N. These modes satisfy the following orthogonality conditions

$$x_j^n\cdot x_j^m=0\ , \qquad n\neq m$$
$$[\hat{V}_{ijk}Q_k^F]\ x_j^m x_n^i=\delta_{mn} \quad (5)$$

From the critical state, the post buckling path will be followed by perturbation techniques. We select a suitable generalized coordinate as perturbation parameter (say q_1) and we write the load Λ and the remaining generalized coordinates q_i ($i \neq 1$) as

$$\Lambda=\Lambda^C+\Lambda^{(1)C}q_1+\frac{1}{2}\Lambda^{(2)C}q_1^2+\ldots$$
$$q_i=q_i^{(1)C}q_1+\frac{1}{2}q_i^{(2)C}q_1^2+\ldots \quad (6)$$

where superscripts (1)C and (2)C indicate first and second order derivatives with respect to the perturbation parameter, evaluated at the critical state.

We next assume that there are two eigenvalues that are relatively close, or even coincident. We refer to these two eigenvalues as *active*, and they play a crucial role in the interactive buckling process. The remaining N-2 eigenvalues (for n = 3,..., N) are *passive*, and they play no role in the analysis.

As a first approximation, we introduce a linear combination of the active modes that may be sufficient to reflect mode interaction given as

$$q_i^I=x_i^k\ \xi_k$$

where, i = 1,..., N and k = 1,2 and ξ_k are *modal amplitudes* (i.e. scalars). The interaction between the $\mathbf{x}^k_i$ modes originates new coupled modes, which will be denoted as $\mathbf{x}_i^{kl}$, also called higher order fields (arising from the interaction between modes $\mathbf{x}^k_i$ and $\mathbf{x}^l_i$).

If we restrict the analysis to a quadratic combination, the initial post-critical path q_i may be described as

$$q_i = q_i^I + q_i^{II} = x_i^k\ \xi_k + x_i^{kl}\ \xi_k\ \xi_l \quad (7)$$

This combination may be found, for example, in the work of Reis (1977), and Reis and Roorda (1979). For a fixed value of k, $\mathbf{x}_i^k$ is a vector of dimension N (an isolated eigenvector); while $\mathbf{x}_i^{kl}$ is a matrix of dimension Nx2, since it reflects the interaction between mode k and all the remaining modes. If we fix k and l, then $\mathbf{x}_i^{kl}$ is a vector of dimension Nx1.

The coupled modes should be orthogonal to the isolated modes, *ie*

$$[V_{ijk}Q_k^F]\ x_j^m\ x_i^{np}=0$$

for m, n, p = 1,2. An additional property of the coupled modes is that of symmetry,

$$x_i^{kl} = x_i^{lk}$$

Notice that the modes considered in the present analysis need not to be simultaneous, meaning that they do not need to correspond to coincident critical loads. Substituting Eq (7) into the W expression (3), after some manipulation, and keeping only up to fourth order terms, we get

$$W=\frac{1}{2!}(\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F)(q_i^I q_j^I+q_i^{II}q_j^{II}+2q_i^I q_j^{II})$$
$$+\frac{1}{3!}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)(q_i^I q_j^I q_k^I+3q_i^{II}q_j^I q_k^I)$$
$$+\frac{1}{4!}(\hat{V}_{ijkl})(q_i^I q_j^I q_k^I q_l^I)$$

We can recognize that some terms vanish due to the orthogonality condition between modes. Hence, the energy W can be written as

$$W=\frac{1}{2!}\delta_{st}\xi_s\xi_t(\Lambda-\Lambda_c^s)$$
$$+\frac{1}{2!}(\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F)x_i^{st}x_j^{uv}\xi_s\xi_t\xi_u\xi_v$$
$$+\frac{1}{3!}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^s x_j^t x_k^u\xi_s\xi_t\xi_u$$
$$+\frac{1}{2}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^{st}x_j^u x_k^v\xi_s\xi_t\xi_u\xi_v$$
$$+\frac{1}{4!}(\hat{V}_{ijkl})x_i^s x_j^t x_k^u x_l^v\xi_s\xi_t\xi_u\xi_v \tag{8}$$

becoming a function of ξ_s, $\mathbf{x}_i^{st}$, and s, t, u, v = 1,2. All the terms depend on ξ_s, but the first, third and fifth term are independent of the higher order fields $\mathbf{x}_i^{st}$. The modes arising from coupling between isolated modes also define paths of equilibrium in the Λ-q_i space. To obtain the mode shapes $\mathbf{x}_i^{st}$ of the coupled modes, we note that the first variation of W with respect to $\mathbf{x}_i^{st}$ is zero, for constant values of the amplitude parameter ξ_s. Thus,

$$\frac{\partial W}{\partial(x_i^{st})}=0=[(\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F)x_j^{uv}$$
$$+\frac{1}{2}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_j^u x_k^v]\xi_s\xi_t\xi_u\xi_v$$

From the above equation we conclude that

$$[\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F]x_j^{uv}=-\frac{1}{2}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_j^u x_k^v \tag{9}$$

The member on the right side is clearly non-zero; this shows that $\mathbf{x}_j^{uv}$ are not eigenvalues of the tangent matrix

$$\left[\hat{V}_{ij}+\Lambda\,\hat{V}_{ijk}\,Q_k^F\right]$$

This is correct, since we have already calculated all the eigenvectors, and denoted them as $\mathbf{x}^s$. Notice that the coupled modes are afunction of Λ.

Let us consider the number of systems (Eq 9) to be solved. If there are M interacting modes, then there will be ½(M+M²) systems to obtain the coupled modes.

The above condition (Eq 9) may be written for simplicity as

$$K_{ij}^T(\Lambda)\;x_j^{uv}=-\alpha_i^{uv}-\Lambda\beta_i^{uv} \tag{10}$$

where,

$$K_{ij}^T(\Lambda)\equiv\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F$$
$$\alpha_i^{uv}\equiv\frac{1}{2}\hat{V}_{ijk}x_j^u x_k^v$$
$$\beta_i^{uv}\equiv\frac{1}{2}\hat{V}_{ijkl}Q_l^F x_j^u x_k^v$$

We write the solution of Eq (10) in the form

$$x_j^{uv} = z_j^{uv} + \Lambda y_j^{uv} \tag{11}$$

By substituting Eq (11) into Eq (10), the following conditions are obtained:

$$K_{ij}^T(\Lambda)\;z_j^{uv}=-\alpha_i^{uv}$$
$$K_{ij}^T(\Lambda)\;y_j^{uv}=-\beta_i^{uv} \tag{12}$$

Equations (12) are valid for all values of Λ; however we are interested in the initial post-critical behavior, so that Λ will be restricted to values $\Lambda_c^u \le \Lambda \le \Lambda_c^v$.

Next, we consider that the interacting modes are associated to near-coincident loads, and with negligible error we may evaluate Eqs (12) at the lowest of the two interacting loads. If we assume that $\Lambda_c^u \le \Lambda_c^v$, then we should set $\Lambda = \Lambda_c^u$ in Eqs. (12). Having chosen $\Lambda = \Lambda_c^u$ means that the matrix $K_{ij}^T|_c$ is singular, and the solution of Eqs (12) requires some additional conditions, which may be taken as:

$$\hat{V}_{ijk}Q_k^F\;z_i^{uv}\;x_j^u=0\quad,$$
$$z_1^{uv}=0$$
$$\hat{V}_{ijk}Q_k^F\;y_i^{uv}\;x_j^u=0$$
$$y_1^{uv}=0 \tag{13}$$

where there is no summation on u in Eqs (13). The second and fourth constraints indicate that if the perturbation parameter chosen to follow the path associated to Λ_c^u is q_1^u, then $\mathbf{z}_1^{uv} = 0$ and $\mathbf{y}_1^{uv} = 0$. The solution of Eq (9) can now be obtained in terms of $\mathbf{x}_j^{uv}$.

We now return to Eq (8) for the energy W. Since the coupled modes $\mathbf{x}_j^{uv}$ are now known, the complete path could be traced if we calculate the mode amplitudes ξ_s. In other words, we have constructed a basis of vectors (the modes $\mathbf{x}_j^s$ and the coupled modes $\mathbf{x}_j^{st}$) to express the equilibrium condition, and we now seek to find how these amplitudes change along the evolving path. To obtain the values of ξ_s we note that at an equilibrium state W must be stationary with respect to all possible variations of the generalized coordinates q_i. Using Eq (7), the variation is written in terms of ξ_s

$$\frac{\partial W[\xi_r,\Lambda]}{\partial \xi_s}=W_{,s}=0$$

for s=1,2 or using Eq (8),

$$W_{,s}=0=[(\Lambda-\Lambda_c^s)\delta_{st}]\xi_t$$
$$+[\frac{1}{2}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^s x_j^t x_k^u]\xi_t\xi_u$$
$$+[2(\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F)x_i^{st}x_j^{uv}$$
$$+2(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^{st}x_j^u x_k^v$$
$$+\frac{1}{6}(\hat{V}_{ijkl})x_i^s x_j^t x_k^u x_l^v]\xi_t\xi_u\xi_v$$

This leads to the following two equilibrium equations in modal space

$$(\Lambda-\Lambda_s^C)\delta_{st}\xi_t+A^{stu}\xi_s\xi_u+B^{stuv}\xi_t\xi_u\xi_v=0 \qquad (14)$$

for s, t, u, v = 1,2. Eq (14) can be written explicitly as

$$(\Lambda-\Lambda_c^1)\xi_1+A^{111}\xi_1^2+2A^{112}\xi_1\xi_2+A^{122}\xi_2^2+$$
$$+B^{1111}\xi_1^3+3B^{1122}\xi_1\xi_2^2+3B^{1112}\xi_1^2\xi_2+B^{1222}\xi_2^3=0$$
$$(\Lambda\Lambda_c^2)\xi_2+A^{211}\xi_1^2+2A^{212}\xi_1\xi_2+A^{222}\xi_2^2+$$
$$+B^{2111}\xi_1^3+3B^{2221}\xi_1\xi_2^2+3B^{2211}\xi_1^2\xi_2+B^{2222}\xi_2^3=0 \qquad (15)$$

where

$$A^{stu}=\frac{1}{2}(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^s x_j^t x_k^u$$
$$B^{stuv}=2(\hat{V}_{ij}+\Lambda\hat{V}_{ijk}Q_k^F)x_i^{st}x_j^{uv}+$$
$$+2(\hat{V}_{ijk}+\Lambda\hat{V}_{ijkl}Q_l^F)x_i^{st}x_j^u x_k^v+\frac{1}{6}\hat{V}_{ijkl}x_i^s x_j^t x_k^u x_l^v \qquad (16)$$

Thus, we have two cubic equations (15) in two unknowns (ξ_1 and ξ_2). The solutions are a function of Λ through the coefficients given in Eq (16).

DISCRETIZED SYSTEM

The structure is discretized using nine-node Lagrangean plate elements based on first order shear deformation theory. The element has six degrees of freedom per node and allows for the modeling of plate assemblies arbitrarily oriented in space.

The strains are written as the summation of a linear plus a n-linear part, in the form

$$\{\epsilon\} = \{\epsilon_0\} + \{\epsilon_1\}$$
$$= \begin{Bmatrix} u_{,x} \\ v_{,y} \\ u_{,y}+v_{,x} \\ -\theta_{x,x} \\ -\theta_{y,y} \\ -\theta_{x,y}-\theta_{y,x} \\ w_{,y}-\theta_{y} \\ w_{,x}-\theta_{x} \\ \theta_{z} \end{Bmatrix} + \frac{1}{2}\begin{Bmatrix} v_{,x}^2+w_{,x}^2 \\ u_{,y}^2+w_{,y}^2 \\ 2w_{,x}w_{,y} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (17)$$

containing three in-plane strains (ϵ_x, ϵ_y, γ_{xy}) at the mid-surface, three curvatures (κ_x, κ_y, κ_{xy}), two out of plane shear strains (γ_{yz} and γ_{zx}), and the in-plane rotation θ_z. Note that Eq (17) contains not simply the von-Karman equations, but also the terms $\partial v/\partial x$ and $\partial u/\partial y$, which are included as in the work of Benito and Sridharan (1985). These additional terms are necessary to represent correctly the nonlinearities at the junction of flange and web of structural members, mainly for the modal interaction analysis.

The linear strain matrix $[B_o]$ is formulated, and also the nonlinear strain matrix $[B_1(q)]$, which is a function of the nodal displacements q. We can further write the non-linear part of the strains as $\{\epsilon_1\}$ = ½ [A] [θ] = $[B_1(q)]$\{g\}, where the matrices [A] and [θ] are still a function of the nodal displacements. The matrix [θ] can be written as [θ] = [G]\{q\}. Next, we write d$\{\epsilon_1\}$ = [A][G]d\{q\}, from which it follows that $[2B_1]$ = [A][G]. On the other hand, the matrix [A(q)] can be written as the product of two matrices, the first containing the derivatives of the shape functions and the second the nodal displacements.

The total potential energy W of the plate, subject to in-plane and transverse loading, can now be written as

$$V = \frac{1}{2}\int\{\sigma\}^T\{\epsilon\}dv - \Lambda\{q\}^T\{f\}$$

where the load vector \{f\} is assumed to be incremented by a single load factor Λ. The stress vector \{σ\} is, in this case, $\{\sigma\} = \{N_x, N_y, N_{xy}, M_x, M_y, M_{xy}, Q_y, Q_x, M_z\}^T$, where N_x, N_y, and N_{xy} are the in-plane stress resultants, M_x, M_y, and M_{xy} are the moment resultants, Q_y and Q_x are the out-of-plane shear stress resultants and M_z is the in-plane moment.

For the case of a plate made of laminated FRP material, the constitutive law is given by Raftoyiannis (Eq 5.15, 1994). The plate stiffness A_{ij}, B_{ij} and D_{ij} are computed using Classical Lamination Theory, and a very small number C^* is used to represent the in-plane rotational stiffness (θ_z).

Materials with non-vanishing bending-extension coupling terms B_{ij} are modeled here with an approximate linear fundamental path. The limitations of this approach are currently being investigated.

The first step in the analysis is to obtain the linear fundamental path $\{Q^F\}$; this is done by solving the system $[K_o]\{Q^F\} = \{P\}$. The second step is the detection of the critical states along the fundamental path, which is done by solving the classical eigenvalue problem, to obtain the critical load Λ^C and the corresponding eigenvector $\{x\}$. Next, attention is given to the study of the post critical path passing through the bifurcation point. We must determine whether the bifurcation is symmetric or asymmetric. For that, it is necessary to compute the matrix $[D_1(x)]$ contracted by the eigenvector $\{x\}$, which can be written in terms of finite element matrices as

$$[D_1(x)] = \int_V \{[2B_1^i(\delta_j)]^T[C][B_0+2\Lambda B_1(Q^F)]\{x\} +[2B_1(x)]^T[C][B_0+2\Lambda B_1(Q^F)] +[B_0+2\Lambda B_1(Q^F)]^T[C][2B_1(x)]\}dv \quad (18)$$

where δ_i is the Kroneker delta, with values $\delta_i = 1$, if $i = j$, and $\delta_i = 0$, if $i \neq j$. The matrix $[2B_1^i(\delta_j)]$ can be computed from $[2B_1] = [A][G]$. The coefficient C may now be computed as

$$C = \{x\}^T [D_1(x)]\{x\}$$

If $C = 0$, the critical state is a symmetric bifurcation, while for $C \neq 0$ the bifurcation is asymmetric. To follow the post buckling path, a perturbation analysis is carried out from the critical state. The variables q_i and Λ are expanded in terms of a perturbation parameter q_1, as indicated in Eq (6). In symmetric bifurcation, the slope $\Lambda^{(1)C}$ of the post buckling path at the bifurcation point vanishes, and $\{q^{(1)C}\} \equiv \{x\}$. As a next step, the $\{q^{(2)C}\}$ coefficients in Eq (6) are computed from

$$[K_T]\ \{q^{(2)C}\} = -[D_1(x)]\{x\}|^C$$

where $[K_T] = [K_o] + \Lambda[K_\sigma]$ is the tangent stiffness matrix at the critical point. The value of one of the components in the vector of the second derivatives of the displacements has to be chosen (*ie*, $q_1^{(2)C} = 0$ if the perturbation parameter is q_1). Finally, the curvature of the post buckling path at the bifurcation point results in

$$\Lambda^{(2)C} = -\frac{\{x\}^T[D_2(x,x)]\{x\}+3\{x\}^T[D_1(x)]\{q^{(2)C}\}}{3\{x\}^T[K_o]\{x\}}$$

The matrix $[D_2(x,x)]$ contracted by the $\{x\}$ eigenvector can be computed from

$$[D_2(x,x)] = \int_V \{[2B_1^i(\delta_j)]^T[C][2B_1(x)]\{x\} + 2[2B_1(x)]^T[C][2B_1(x)]\}dv$$

The coefficients A^{stu} (Eq 16) of the quadratic terms in the equilibrium equations are related to the contracted matrix $[D_1(\mathbf{x})]$ in the following manner

$$A^{stu}=\frac{1}{2}x_i^s[D_1(x_j^t)]x_k^u$$

Since the contracted matrix $[D_1(\mathbf{x})]$ is afunction of the load parameter Λ, we can write

$$A^{stu}=\frac{1}{2}x_i^s\{[E_1(x_j^t)]+\Lambda[E_2(x_j^t)]\}x_k^u$$

where

$$[E_1(x)]=\int\{[2B_1(\delta)]^T[C][B_0]\{x\} +[2B_1(x)]^T[C][B_0]+[B_0]^T[C][2B_1(x)]\}dv$$

$$[E_2(x)]=\int\{[2B_1(\delta)]^T[C][2B_1(q^F)]\{x\} +[2B_1(x)]^T[C][2B_1(q^F)] +[2B_1(q^F)]^T[C][2B_1(x)]\}dv \quad (19)$$

or

$$A^{stu}=F^{stu}+\Lambda G^{stu}$$

where

$$F^{stu}=\frac{1}{2}x_i^sE_1(x_j^t)x_k^u$$

$$G^{stu}=\frac{1}{2}x_i^sE_2(x_j^t)x_k^u$$

We will now be concerned with the coefficients of the cubic terms in the equilibrium equations (14). Since they contain the second order field, we must first solve the system (12) and define the relation (11). The vectors α_i^{uv} and β_i^{uv} can be written in matrix form as

$$\alpha^{uv}=\frac{1}{2}[E_1(x^u)]\{x^v\}$$

$$\beta^{uv}=\frac{1}{2}[E_2(x^u)]\{x^v\}$$

where the contracted matrices $[E_1]$ and $[E_2]$ are defined in Eq (19). Hence, the system (12) can be solved along with the conditions (13) for the vectors α_i^{uv} and β_i^{uv}. The coefficients B^{stuv} are written in matrix form as

$$B^{stuv}=2([K_0]+\Lambda[K_\sigma]) \times(\{z^{st}\}+\Lambda\{y^{st}\})(\{z^{uv}\}+\Lambda\{y^{uv}\}) +2([E_1(x^s)]+\Lambda[E_2(x^s)])(\{z^{tu}\} +\Lambda\{y^{tu}\})\{x^v\} +\frac{1}{6}[D_2(x^s,x^t)]\{x^u\}\{x^v\}$$

Expanding the expression (40) and collecting the terms in like-powers of the load parameter Λ, we can write

$$B^{stuv}=M^{stuv}+\Lambda N^{stuv}+\Lambda^2 P^{stuv}+\Lambda^3 Q^{stuv}$$

w[illegible]

$$M^{stuv} = 2\{z^{st}\}^T[K_0]\{z^{uv}\}+2\{z^{st}\}^T[E_1(x^{u})]\{x^{v}\} + \frac{1}{6}\{x^{s}\}^T[D_2(x^{t},x^{u})]\{x^{v}\}$$

$$N^{stuv} = 2\{z^{st}\}^T[K_\sigma]\{z^{uv}\}+2\{z^{st}\}^T[K_0]\{y^{uv}\} +2\{y^{st}\}^T[K_0]\{z^{uv}\}+2\{y^{st}\}^T[E_1(x^{u})]\{x^{v}\} +2\{z^{st}\}^T[E_2(x^{u})]\{x^{v}\}$$

$$P^{stuv} = 2\{y^{st}\}^T[K_0]\{y^{uv}\}+2\{z^{st}\}^T[K_\sigma]\{y^{uv}\} +2\{y^{st}\}^T[K_\sigma]\{z^{uv}\}+2\{y^{st}\}^T[E_2(x^{u})]\{x^{v}\}$$

$$Q^{stuv} = 2\{y^{st}\}^T[K_\sigma]\{y^{uv}\}$$

and s, t, u, v = 1,2. The energy terms are also symmetric in this case, therefore

$$A^{stu}=A^{tus}=A^{ust}$$
$$B^{stuv}=B^{tuvs}=B^{uvst}=B^{vstu}$$

Finally, Eqs (14) are completely defined and can be solved using perturbation techniques or using the Newton-Raphson technique.

The imperfection sensitivity of the system can be investigated by using the imperfection parameters ξ_i^{imp}, that repres[illegible] the amplitude of an imperfection with the shape of the m[illegible]e $\mathbf{x}^i$. The equilibrium equations for the imperfect case of two interacting modes become

$$(\Lambda-\Lambda_c^s)\delta_{st}\xi_t+A^{stu}\xi_t\xi_u+B^{stuv}\xi_t\xi_u\xi_v=\Lambda\xi_t^{imp} \quad (20)$$

where s, t, u, v = 1,2. The Newton-Raphson technique is used in the next section to solve these equations.

NUMERICAL RESULTS

To illustrate the numerical procedure, we consider a Fiber Reinforced Plastic (FRP) I-column produced by pultrusion. The geometry of the column is defined by b = 6 in; h = 6 in, and L = 100 in. The material properties for the flanges are A_{11} = 893,500 lb/in; A_{22} = 343,000 lb/in; A_{12} = 130,800 lb/in; A_{66} = 113,600 lb/in; D_{11} = 4,289 lb in; D_{22} = 2,029 lb in; D_{12} = 807.3 lb in; D_{66} = 641.5 lb in, A_{44} = A_{55} = 94,670 lb/in; and A_{45} = A_{54} = 0. For the web the material properties are A_{11} = 893,500 lb/in; A_{22} = 343,000 lb/in; A_{12} = 130,800 lb/in; A_{66} = 113,600 lb/in; D_{11} = 4,090 lb in; D_{22} = 1,863 lb in; D_{12} = 731.6 lb in; D_{66} = 596.1 lb in, A_{44} = A_{55} = 94,670 lb/in; and A_{45} = A_{54} = 0.

The column is discretized using 45 elements. The boundary conditions applied to the FE model are as follows: t[illegible] [illegible]nds of the column are simply supported, with free rotation about the weak axis of the cross section. Hence, a global mode (Euler) is expected to occur about the weak axis.

Table 1. Critical modes and corresponding critical loads for the pultruded I-column

Mode	Cr. Load
1231	-193,994.50
1232	-191,354.23
1233	-189,244.32
1234	-171,387.39
1235	-175,405.49
1236	-174,636.60
1237	-196,773.89
1238	-198,689.01
1239	-179,645.15
1240	-160,458.93
1241	-142,923.56
1242	-27,323.60
1243	-27,461.80
1244	-29,048.20
1245	-29,432.43
1246	-29,761.56
1247	-31,698.91
1248	-33,527.09
1249	-142,770.19
1250	-36,572.63
1251	-129,134.71
1252	-122,922.03
1253	-40,905.58
1254	-36,342.64
1255	-44,589.87
1256	-55,340.84
1257	-38,960.76
1258	-60,111.63
1259	-40,119.62
1260	-42,739.29
1261	-43,760.90
1262	-47,486.69
1263	-117,812.37
1264	-131,584.77
1265	-115,168.30
1266	-128,594.97

Perfect system: The model is first solved for pre-buckling displacements. The eigenvalue problem is then formulated to solve for the critical buckling loads and the associated eigenmodes. A selected number of critical loads that are the eigenvalues of the buckling problem are shown in Table 1.

A mode number is assigned to each eigenvalue with reference to Fig 1, which contains schematics of the modal shapes associated to the eigenvalues presented in Table 1. Each mode is represented by the cross-sectional modal deformation at the x = L/2, the lateral deformation of the longitudinal axis passing through the centroid for the entire length of the column, and the lateral deformation of the right tip of the top flange for the entire length as well.

The lowest eigenvalue (Mode #1242 in Table 1) corresponds to the local buckling mode, and is Λ_1^C = -27,323 lb. The associated eigenmode $\mathbf{x}^1$ can be seen in Fig 1. Both the flanges rotate and the web bends with eight waves along the length of the column.

stable symmetric (Fig 2), and the curvature is found to be $\Lambda_1^{(2)C}$ = -172,182 lb/in². The global mode $\mathbf{x}^2$ (Euler mode about the weak axis) is next identified from Figure 1 (Mode #1246). The cross-section remains undeformed, but the whole beam buckles with one wave in the global sense. The critical load is Λ_2^C = -29,761 lb (Table 1). The post buckling path is also stable and symmetric, and the curvature is computed as $\Lambda_2^{(2)C}$ = -211 lb/in² that is a low value, compared to the one of the local post buckling path.

Next, the eigenvectors of the two selected modes are normalized according to Eq (5), where the highest components are 4.48 for the global mode $\mathbf{x}^1$ and 1.6755 for the local mode $\mathbf{x}^2$. The coefficients F^{stu}, G^{stu}, M^{stuv}, and N^{stuv} (where s, t, u, v = 1,2) of the two equilibrium equations in modal space are then computed. The numerical values of the interaction coefficients that are different from zero are: M^{1111} = 324,516.24; M^{1112} = -6,174.24; M^{1122} = 71,751.16; M^{1222} = 895.02; M^{2222} = 74,743.75; N^{1111} = 0.15383; N^{1112} = -0.18518; N^{1122} = 5.15215; N^{1222} = 0.02594 and N^{2222} = 0.02651. Hence, the following equilibrium equations are defined, and can be solved for specified values of the load parameter.

$$(\Lambda+27323)\xi_1+(324516.24+0.15383\Lambda)\xi_1^3 -(18522.72+0.55554\Lambda)\xi_1^2\xi_2 +(215253.48+15.45645\Lambda)\xi_1\xi_2^2 +(895.02+0.02594\Lambda)\xi_2^2=0$$

$$(\Lambda+29761)\xi_2-(6174.24+0.18518\Lambda)\xi_1^3 +(215253.48+15.45645\Lambda)\xi_1^2\xi_2 +(2685.06+0.07782\Lambda)\xi_1\xi_2^2 +(74743.75+0.02651\Lambda)\xi_2^3=0$$

Notice that by keeping only the first order field, the values of the coefficients become M^{1111} = 2,701,272; M^{1222} = -2,500; M^{1122} = 64,993; M^{1222} = -210.6 and M^{2222} = 174,216; while all N^{stuv} become zero. Then, the system is imperfection insensitive, and the interaction phenomenon cannot be retrieved. In order to study the interaction between the two modes, we see that the coefficients which couple the two modes are of significant importance.

Next, in order to determine the tertiary (coupled) path, we expect the system to experience loss of stability. This means that the tertiary path will be branching from the lower secondary path, which corresponds to the local mode. A Newton-Raphson technique for solving the equilibrium equations is used here. Hence, the bifurcation point is detected for load $\Lambda^{(0)}$ = -28,725 and amplitude of the primary local mode $\xi_1^{(0)}$ = 0.06681. The projection of the tertiary path on Λ-ξ_1 plane is plotted in Fig 5. We see that on the tertiary path Λ-ξ_1, the load carrying capacity decreases (loss of stability) for increasing ξ_1 (local mode amplitude).

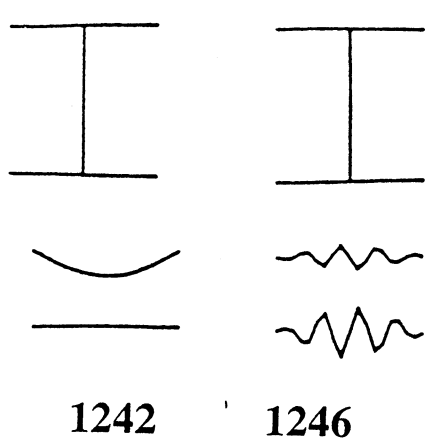

Fig 1. Modal shapes for the I-column

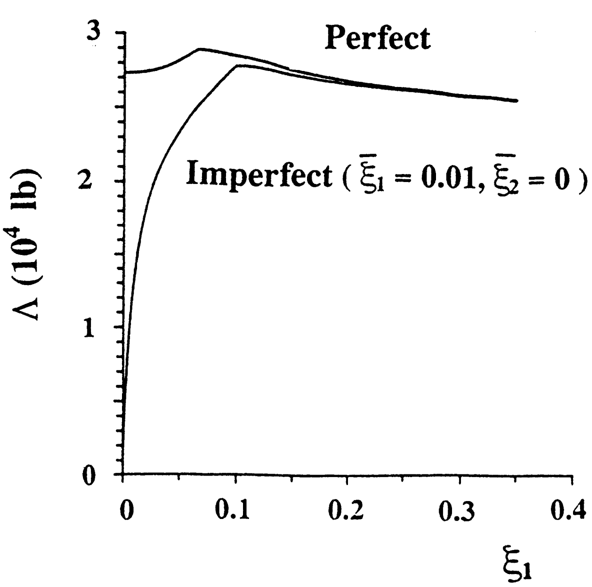

Fig 2. Equilibrium path, projection on the Λ-ξ_1 plane (local)

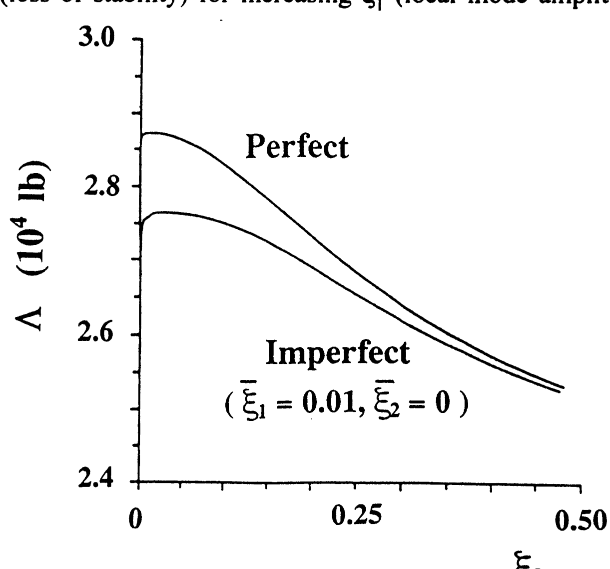

Fig 3. Equilibrium path, projection on the Λ-ξ_2 plane (global)

The projection on the Λ-ξ_2 plane (ξ_2 being the amplitude of the global mode) is shown in Fig 3. A comparison between the tertiary path obtained by a three-parameter Ritz approxi- e solution of the same problem (Godoy *et al*, 1994) and the present finite element solution is shown in Fig 4. Thedifference in the results are caused by the approximation to the boundary conditions assumed in the Ritz model.

Imperfect system: We now consider the I-column having in the unloaded state an initial imperfection similar to one of the buckled mode shapes, global or local. The imperfection amplitude is defined by the parameter ξ_i^{imp}. Then, the equilibrium equations (20) hold for the imperfect system and are expressed in the modal space. These equations are solved by the Newton-Raphson method.

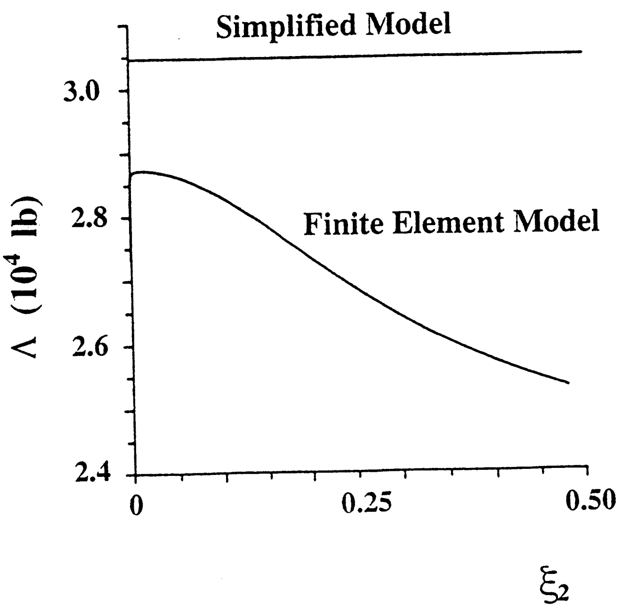

Fig 4. Comparison between simple and FE model (equilibrium path projection on Λ-ξ_2, global)

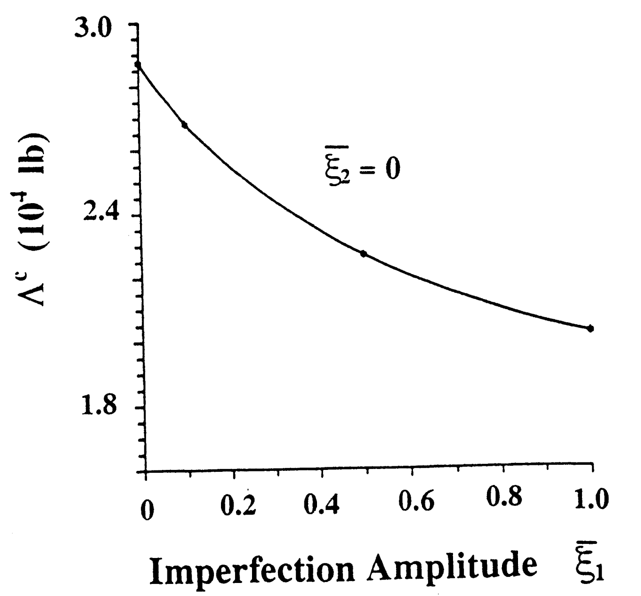

Fig 5. Imperfection sensitivity curves

The imperfection sensitivity curves for two mode interaction is shown in Figure 5. It can be seen that the I-column has low imperfection sensitivity, *ie* for a significantly imperfect column, the reduction of load carrying capacity is about 30%.

CONCLUSIONS

Fiber reinforced plastic structural shapes are shown to exhibit buckling mode interaction. Because of the linearity of the material up to failure, and the linearity of the fundamental path, the W-formulation was successfully applied to study the post-buckling behavior interms of two-mode interaction.

Higher order fields were necessary to detect mode interaction, and these fields were constructed from the information contained in the W-energy of the system. A numerical procedure, based on the finite element method and perturbation techniques, was developed for post-buckling analysis of structures modeled as plate assemblies.

The effect of two interacting modes on the post-buckling behavior was accounted by studying the equilibrium paths in the reduced modal space of the active coordinates. The resulting two non-linear equations describe the complete behavior of the system in the vicinity of the equilibrium paths. These equations were conveniently solved by the Newton-Raphson technique, in order to track the perfect primary, secondary, and tertiary paths, the latter arising from mode interaction. Both the perfect and the imperfect systems were formulated and solved, the latter having imperfections in the shape of one or two the interacting modes. By using a finite element discretization as plate assemblies, all buckling modes (local and global) as well wave amplitude modulation are automatically taken into account. A fiber reinforced compositecolumn was analyzed and the results favorably agree with the results of a simplified model. While all the isolated mode secondary paths were found to be stable, the column was found to be imperfection sensitive once mode interaction was acknowledged.

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