THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS
345 E. 47th St., New York, N.Y. 10017

95-APM-10

Analytical Expressions for the Relaxation Moduli of Linear Viscoelastic Composites With Periodic Microstructure

R. Luciano[1]
University of Cassino,
via Zamosh 43,
Cassino, Italy

E. J. Barbero
West Virginia University,
Morgantown, WV 26506-6106,
Assoc. Mem. ASME.

In this paper the viscoelastostatic problem of composite materials with periodic microstructure is studied. The matrix is assumed linear viscoelastic and the fibers elastic. The correspondence principle in viscoelasticity is applied and the problem in the Laplace domain is solved by using the Fourier series technique and assuming the Laplace transform of the homogenization eigenstrain piecewise constant in the space. Formulas for the Laplace transform of the relaxation functions of the composite are obtained in terms of the properties of the matrix and the fibers and in function of nine triple series which take into account the geometry of the inclusions. The inversion to the time domain of the relaxation and the creep functions of composites reinforced by long fibers is carried out analytically when the four-parameter model is used to represent the viscoelastic behavior of the matrix. Finally, comparisons with experimental results are presented.

Introduction

A large number of micromechanical models have been developed to estimate the elastic properties of composite materials (see Christensen, 1990; Mura, 1987; Nemat Nasser and Hori, 1993). However, few theoretical and experimental results are available in the field of viscoelastic behavior of heterogeneous media.

The first micromechanical model used to evaluate the macroscopic viscoelastic properties of fiber-reinforced materials was the cylinder assemblage model proposed by Hashin (1965, 1966), where the analogy between the elastic and the viscoelastic relaxation moduli of heterogeneous materials with identical phase geometry was presented. This analogy is known as the correspondence principle (Christensen, 1979) and many authors applied it. For example, Christensen (1969) proposed an approximate formula for the effective complex shear modulus in the case of materials with two viscoelastic phases by using the composite sphere model.

Laws and McLaughlin (1978) estimated the viscoelastic creep compliances of several composites by applying the self-consistent method. They used Stieltjes convolution integrals to formulate the problem in the Carson domain and a numerical inversion method to obtain the solution in the time domain. Yancey and Pindera (1990) estimated the creep response of unidirectional composites with linear viscoelastic matrices and elastic fibers by applying the micromechanical model proposed by Aboudi (1991) to obtain the Laplace transform of the effective viscoelastic moduli. Then, they used Bellman's numerical method for the inversion to the time domain. For different geometry of the inclusions, Wang and Weng (1992) adopted the Eshelby-Mori-Tanaka method (Mori and Tanaka, 1973) in order to obtain the overall linear viscoelastic properties of the corresponding composite material.

Finally, it is possible to conclude that many micromechanical models applied for the analysis of the elastic behavior of composites have been extended to the viscoelastic case. However, no theory has been developed for linear viscoelastic solids with periodic microstructure, even though many results are available for the elastic case (Nemat-Nasser and Taya, 1981, 1986; Nemat-Nasser et al., 1982; Nemat-Nasser and Hori, 1993). For this reason, in the present paper, close-form expressions in the Laplace domain for the coefficients of the linear viscoelastic relaxation tensor of composite materials with periodically distributed elastic inclusions and linear viscoelastic matrix are proposed. Moreover, the inversion to the time domain is carried out *analytically* for composites reinforced by long fibers and when the viscoelastic behavior of the matrix can be represented by a four-parameter model.

[1]Visiting Assistant Professor, Department of Mechanical and Aerospace Engineering, West Virginia University, Morgantown, WV 25606-6106.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for presentation at the Joint Applied Mechanics and Materials Summer Meeting of the ASME, Los Angeles, CA, June 28–30, 1995.

Discussion on this paper should be addressed to the Technical Editor, Professor Lewis T. Wheeler, Department of Mechanical Engineering, University of Houston, Houston, TX 77204-4702, and will be accepted until four months after final publication of the paper itself in the ASME JOURNAL OF APPLIED MECHANICS.

Manuscript received by the ASME Applied Mechanics Division, Mar. 16, 1994; final revision, Dec. 2, 1994. Associate Technical Editor: G. J. Drorak.

Paper No. 95-APM-10.

Copies of this paper will be available until Sept. 1996.

More complex creep behaviour of the matrix requires numerical inversion to the time domain (Barbero and Luciano 1995). Finally, comparisons with available experimental data obtained by Skudra and Auzukalns (1973) are presented.

Viscoelastic Constitutive Equations

The constitutive equations of a linear viscoelastic isotropic material can be expressed in the time domain in the following way:

$$\sigma(t) = I^{(2)}\int_{-\alpha}^{t} \lambda(t-\tau)tr\dot{\epsilon}(\tau)d\tau + 2\int_{-\alpha}^{t} \mu(t-\tau)\dot{\epsilon}(\tau)d\tau, \quad (1)$$

where $\sigma(t)$ and $\epsilon(t)$ are the stress and strain tensor, $\lambda(t)$ and $\mu(t)$ are the two stress-relaxation functions, the dot indicates the differentiation with respect to time, and $I^{(2)}$ denotes the identity second-order tensor.

The inverse relations of Eq. (1) can be written in terms of the creep functions $\theta(t)$ and $\zeta(t)$ as

$$\epsilon(t) = I^{(2)}\int_{-\alpha}^{t} \theta(t-\tau)tr\dot{\sigma}(\tau)d\tau + 2\int_{-\alpha}^{t} \zeta(t-\tau)\dot{\sigma}(\tau)d\tau. \quad (2)$$

Let us assume that the relaxation and the creep functions are smooth functions (Gurtin and Sternberg, 1962) and denote the Laplace transform of a function $f(t)$ as

$$\tilde{f}(s) = \int_{0}^{\alpha} f(t)\exp(-st)dt, \quad (3)$$

then the Eqs. (1) and (2) can be expressed in the Laplace domain as

$$\tilde{\sigma}(s) = s\tilde{\lambda}(s)tr\tilde{\epsilon}(s)I^{(2)} + 2s\tilde{\mu}(s)\tilde{\epsilon}(s) = s\tilde{L}(s)\tilde{\epsilon}(s), \quad (4)$$

$$\tilde{\epsilon}(s) = s\tilde{\theta}(s)tr\tilde{\sigma}(s)I^{(2)} + 2s\tilde{\zeta}(s)\tilde{\sigma}(s) = s\tilde{M}(s)\tilde{\sigma}(s), \quad (5)$$

where the Laplace transform of the creep compliance $\tilde{M}(s)$ and the relaxation tensor $\tilde{L}(s)$ satisfy the following relation:

$$\tilde{M}(s) = \frac{1}{s^2}\tilde{L}(s)^{-1}. \quad (6)$$

The Poisson ratio in the transformed domain ν^{TD} is written in terms of $\tilde{\lambda}(s)$ and $\tilde{\mu}(s)$ as

$$\nu^{TD} = \tilde{\lambda}(s)/2\big(\tilde{\lambda}(s) + \tilde{\mu}(s)\big). \quad (7)$$

For simplicity, and consistently with earlier work (Aboudi, 1991; Wang and Weng, 1992), only the set of linear viscoelastic materials whose Poisson ratio remains constant in the course of the deformation (i.e., $\nu(t) = \nu = \nu^{TD}$) will be considered. However, the Poisson ratio of the fibers can be different of that of the matrix.

Periodic Eigenstrain in the Laplace Domain

Suppose that an infinitely extended linearly viscoelastic solid is represented by an assembly of unit cells and let each cell D be a parallelepiped with dimensions a_j in the directions of the coordinate axes x_j where $j = 1, 2, 3$ (see Fig. 1) and let V be its volume. Then, let use denote with Ω the part of D occupied by the inclusions, with $D - \Omega$ the part of D occupied by the matrix, and with v_f be the volume fraction of Ω. The constitutive equations of the linear viscoelastic matrix in the Laplace domain can be written by using Eq. (4) as

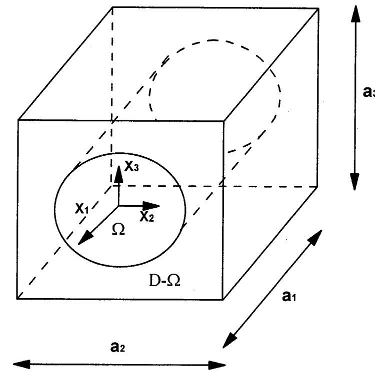

Fig. 1 Geometry of the unit cell D

$$\tilde{\sigma}(s,x) = s\tilde{L}(s)\tilde{\epsilon}(s,x) \text{ in } D - \Omega, \quad (8)$$

while the elastic inclusion is represented as

$$\tilde{\sigma}(s,x) = sL'(s)\tilde{\epsilon}(s,x) = L'\epsilon(s,x) \text{ in } \Omega, \quad (9)$$

and L' is the elastic stiffness tensor of the inclusion. In order to simulate the inclusions inside the body, the equivalent eigenstrain method will be used (see Mura, 1987; Nemat Nasser and Hori, 1993). The idea is to apply an eigenstrain on the homogeneous solid to obtain the equivalence between the stress in the homogeneous material and the heterogeneous one. Then, consider the Laplace transform of the homogenization eigenstrain $\tilde{\epsilon}^*(s,x)$ which must be periodic in x for the particular geometry of the problem and different from zero only in Ω. By using this technique, the inclusion problem is reduced to a viscoelastostatic problem of an homogeneous solid subject to a suitable periodic eigenstrain $\tilde{\epsilon}^*(s,x)$.

Next, by using the correspondence principle for linear viscoelastic solids (see Christensen, 1979; Aboudi, 1991), the relation between the eigenstrain and the strain inside Ω will be introduced in the Laplace domain. Since the material is linear viscoelastic, the Laplace transform of the actual stress tensor $\tilde{\sigma}(s,x)$ inside the unit cell can be expressed in terms of $\tilde{\epsilon}^*(s,x)$ and the Laplace transform of the actual strain tensor $\tilde{\epsilon}(s,x)$ in the following way:

$$\tilde{\sigma}(s,x) = S\tilde{L}(s)\big(\tilde{\epsilon}(s,x) - \tilde{\epsilon}^*(s,x)\big) \text{ for } x \in D, \quad (10)$$

while Eq. (8) is valid in $D - \Omega$. Then, assuming the body forces equal to zero, the tensor $\tilde{\sigma}(s,x)$ must satisfy the following equilibrium conditions:

$$\text{div}\tilde{\sigma}(s,x) = 0 \text{ for } x \in D, \quad (11)$$

where div denotes the divergence of a tensor field.

Since the object of this paper is the analysis of composite materials with periodic microstructure, the eigenstrain $\tilde{\epsilon}(s,x)$ simulates the presence of the periodic inclusions. Furthermore, in a solid with periodic microstructure, the boundary conditions of the unit cell D are governed by the periodicity in x of the microstructure and are satisfied by expanding the displacements and the eigenstrain or their Laplace transforms ($\tilde{\mathbf{u}}(s,x)$ and $\tilde{\epsilon}^*(s,x)$) in the following Fourier series representation:

$$\tilde{\mathbf{u}}(s,x) = \sum_{\xi}^{\pm\infty} \bar{\tilde{\mathbf{u}}}(s,\xi)\exp(i\xi x), \quad (12)$$

$$\tilde{\epsilon}(s,x) = \text{sym}(\nabla\tilde{\mathbf{u}}(s,x)) = \sum_{\xi}^{\pm\infty} \bar{\tilde{\epsilon}}(s,\xi)\exp(i\xi x), \quad (13)$$

$$\tilde{\epsilon}^*(s,x) = \sum_{\xi}^{\pm\infty} \bar{\tilde{\epsilon}}^*(s,\xi)\exp(i\xi x), \quad (14)$$

where $\xi = \{\xi_1, \xi_2, \xi_3\}$ with $\xi_j = 2\pi n_j/a_j$ ($n_j = 0, \pm 1, \pm 2 \ldots . . j$ not summed, $j = 1, 2, 3$), $i = \sqrt{-1}$ and

$$\bar{\tilde{\mathbf{u}}}(s, \xi) = \frac{1}{V}\int_D \tilde{\mathbf{u}}(s, x)\exp(-i\xi x)dx, \quad (15)$$

$$\bar{\tilde{\epsilon}}(s, \xi) = \frac{i}{2}\left[\xi \otimes \bar{\tilde{\mathbf{u}}}(s, x) + \bar{\tilde{\mathbf{u}}}(s, \xi) \otimes \xi\right], \quad (16)$$

$$\bar{\tilde{\epsilon}}^*(s, \xi) = \frac{1}{V}\int_D \tilde{\epsilon}^*(s, x)\exp(-i\xi x)dx. \quad (17)$$

Combining Eq. (10) and Eq. (11) gives

$$\operatorname{div}\left(s\tilde{L}(s)(\tilde{\epsilon}(s, x) - \tilde{\epsilon}^*(s, x))\right) = 0 \text{ in } D. \quad (18)$$

Then, by Eqs. (13), (16), and (14) in Eq. (18), the following expressions are obtained:

$$-\xi \cdot \tilde{L}(s)(\xi \otimes \bar{\tilde{\mathbf{u}}}(s, \xi)) = i\xi \cdot \tilde{L}(s)\bar{\tilde{\epsilon}}^*(s, \xi) \quad \text{for every } \xi \neq 0, \quad (19)$$

where the symbols $\otimes$ and $\cdot$ represent the outer and the inner product, respectively (Spiegel 1959). Thus, since $\tilde{L}(s)$ represents the Laplace transform of the viscoelastic relaxation tensor of the matrix, the coefficients $\bar{\tilde{\mathbf{u}}}\ (s, \xi)$ are obtained uniquely in terms of the $\bar{\tilde{\epsilon}}^*\ (s, \xi)$ in the following way:

$$\bar{\tilde{\mathbf{u}}}(s, \xi) = -i\left(\xi \cdot \tilde{L}(s) \cdot \xi\right)^{-1} \cdot \xi \cdot \tilde{L}(s)\bar{\tilde{\epsilon}}^*(s, \xi) \quad \text{for every } \xi \neq 0, \quad (20)$$

and from Eq. (16), the Fourier coefficients of the corresponding strain are

$$\bar{\tilde{\epsilon}}(s, \xi) = \operatorname{sym}\left(\xi \otimes \left(\xi \cdot \tilde{L}(s) \cdot \xi\right)^{-1} \otimes \xi\right):\tilde{L}(s)\bar{\tilde{\epsilon}}^*(s, \xi) \quad \text{for every } \xi \neq 0. \quad (21)$$

Finally denoting

$$P'(s, \xi) = \operatorname{sym}\left(\xi \otimes \left(\xi \cdot \tilde{L}(s) \cdot \xi\right)^{-1} \otimes \xi\right), \quad (22)$$

the actual strain inside the inclusion from Eq. (21) using Eqs. (13) and (17) is

$$\tilde{\epsilon}(s, x) = \frac{1}{V}\sum_{\xi}^{\pm\infty\prime} P'(s, \xi):\tilde{L}(s)\int_D \tilde{\epsilon}^*(s, x') \times \exp(-i\xi(x' - x))dx' \quad (23)$$

where a prime on the sum indicates that $\xi = 0$ is excluded in the summation.

Since the aim of this work is to obtain the overall viscoelastic properties, the exact expression of the strain tensor $\tilde{\epsilon}(s, x)$ is not necessary. Only its volume average on Ω denoted by $(\overline{\tilde{\epsilon}(s)} = 1/V_\Omega \int_\Omega \tilde{\epsilon}(s, x)dx)$ is needed,

$$\overline{\tilde{\epsilon}(s)} = \frac{1}{V}\sum_{\xi}^{\pm\infty\prime} P'(s, \xi):\tilde{L}(s)\left(\frac{g_0(\xi)}{V_\Omega}\right)\int_D \tilde{\epsilon}^*(s, x') \times \exp(-i\xi x')dx', \quad (24)$$

where V_Ω is the volume of the inclusion and

$$g_0(\xi) = \int_\Omega \exp(i\xi x)dx. \quad (25)$$

In a periodic microstructure, the equivalent eigenstrain is not constant in Ω. However, in order to solve the problem analytically, an approximation of Eq. (24) is introduced using a constant $\tilde{\epsilon}^*(s, x)$. While it is possible to use a polynomial approximation for the eigenstrain, the differences between the two approaches have been shown to be small in the elastic case (Nemat-Nasser and Taya, 1981). Then, replacing $\tilde{\epsilon}^*(s, x)$ with its volume average $\overline{\tilde{\epsilon}^*(s)}$, Eq. (24) becomes

$$\overline{\tilde{\epsilon}(s)} = \frac{1}{V}\sum_{\xi}^{\pm\infty\prime} P'(s, \xi):\tilde{L}(s)\left(\frac{g_0(\xi)g_0(-\xi)}{V_\Omega}\right)\overline{\tilde{\epsilon}^*(s)}, \quad (26)$$

or

$$\tilde{\epsilon}(s) = v_f\sum_{\xi}^{\pm\infty\prime}\left(\frac{g_0(\xi)}{V_\Omega}\right)\left(\frac{g_0(-\xi)}{V_\Omega}\right)P'(s, \xi):\tilde{L}(s):\overline{\tilde{\epsilon}^*(s)}, \quad (27)$$

and by denoting

$$t(\xi) = v_f\left(\frac{g_0(\xi)}{V_\Omega}\right)\left(\frac{g_0(-\xi)}{V_\Omega}\right), \quad (28)$$

and

$$P(s) = \sum_{\xi}^{\pm\infty\prime} t(\xi)P'(s, \xi), \quad (29)$$

the following expression holds:

$$\overline{\tilde{\epsilon}(s)} = P(s):\tilde{L}(s):\overline{\tilde{\epsilon}^*(s)}. \quad (30)$$

Note that Eq. (30) represents the relation between the volume average of the strain inside $\Omega(\overline{\tilde{\epsilon}(s)})$ and the volume average of the applied eigenstrain $(\overline{\tilde{\epsilon}^*(s)})$ in the transformed domain.

Overall Linear Viscoelastic Relaxation Tensor

In order to obtain the homogenization eigenstrain which simulates the presence of the periodic inclusions inside the body, let us consider an applied average strain tensor with Laplace transform $\bar{\tilde{\epsilon}}_0(s)$. Under this condition, the Laplace transform of the average stress in the inclusion is

$$\sigma_{\text{het}}(s) = s\tilde{L}'(s):\left(\bar{\tilde{\epsilon}}_0(s) + P(s):\tilde{L}(s):\overline{\tilde{\epsilon}^*(s)}\right), \quad (31)$$

where σ_{het} indicates the stress in the heterogeneous material. In the equivalent homogeneous solid, the Laplace transform of the average stress σ_{hom} is

$$\sigma_{\text{hom}}(s) = s\tilde{L}(s):\left(\bar{\tilde{\epsilon}}_0(s) + \left(P(s):\tilde{L}(s) - I^{(4)}\right):\overline{\tilde{\epsilon}^*(s)}\right). \quad (32)$$

Then, by imposing the equivalence between the stress in the homogeneous material σ_{hom} and the heterogeneous one σ_{het} (equivalent eigenstrain method), the following average consistency condition in he Laplace domain is obtained (see Nemat-Nasser and Hori, 1990, for the elastic case):

$$\tilde{L}'(s):\left(\bar{\tilde{\epsilon}}_0(s) + P(s):\tilde{L}(s):\overline{\tilde{\epsilon}^*(s)}\right) = \tilde{L}(s):\left(\bar{\tilde{\epsilon}}_0(s) + \left(P(s):\tilde{L}(s) - I^{(4)}\right):\overline{\tilde{\epsilon}^*(s)}\right), \quad (33)$$

where L' is the elastic tensor of the inclusion and $I^{(4)}$ is the identity fourth-order tensor. Observe that the tensor $P(s)$ takes into account the geometry of the inclusion and can be evaluated once and for all. Then from Eq. (33), the equivalent average volume eigenstrain $\overline{\tilde{\epsilon}^*(s)}$ can be solved in terms of the tensors $\tilde{L}'(s)$, $\tilde{L}(s)$, $P(s)$, and $\bar{\epsilon}_0$ for every s as

$$\overline{\tilde{\epsilon}^*(s)} = \left[\left(\left(\tilde{L}(s) - \tilde{L}'(s)\right)^{-1} - P(s)\right)\tilde{L}(s)\right]^{-1}\bar{\tilde{\epsilon}}_0(s). \quad (34)$$

Furthermore, using the linear constitutive equation in the Laplace domain, the Laplace transform of the uniform overall stress $\bar{\tilde{\sigma}}_0(s)$ in the unit cell is

$$s\tilde{L}^*(s):\overline{\tilde{\epsilon}_0}(s) = s\tilde{L}(s):\left(\overline{\tilde{\epsilon}_0}(s) - v_f\overline{\epsilon^*(s)}\right), \quad (35)$$

where $\tilde{L}^*(s)$ is the overall relaxation tensor of the composite material. By using Eq. (34) and noting that $\overline{\epsilon_0}(s)$ is arbitrary, the following expression of $\tilde{L}^*(s)$ is obtained:

$$s\tilde{L}(s) = s\tilde{L}(s) - sv_f\left(\left(\tilde{L}(s) - \tilde{L}'(s)\right)^{-1} - P(s)\right)^{-1}. \quad (36)$$

In particular, if the matrix is isotropic, denoting by $\bar{\xi} = \xi/|\xi|$, the tensor $P(s)$ is (Mura, 1987; Nemat-Nasser and Hori, 1990)

$$P(s) = \frac{1}{\tilde{\mu}_0(s)}\sum_{\xi}^{\pm\infty'} t(\xi)\left(\mathrm{sym}(\bar{\xi}\otimes I^{(2)}\otimes\bar{\xi}) - \frac{1}{2(1-\nu_0)}(\bar{\xi}\otimes\bar{\xi}\otimes\bar{\xi}\otimes\bar{\xi})\right), \quad (37)$$

where $\mu_0(s)$ and ν_0 are the Laplace transform of the shear modulus and the Poisson ratio of the matrix, respectively. Hence, when the matrix and the inclusion are both isotropic, Eq. (36) can be written:

$$s\tilde{L}^*(s) = s\tilde{\lambda}_0(s)I^{(2)}\otimes I^{(2)} + 2s\tilde{\mu}_0(s)I^{(4)} - v_f\Big[\left(s\tilde{\lambda}_0(s) - \lambda_1\right)I^{(2)}\otimes I^{(2)} + 2\left(s\tilde{\mu}_0(s) - \mu_1\right)I^{(4)}\Big)^{-1} + - \frac{1}{s\tilde{\mu}_0(s)}\sum_{\xi}^{\pm\infty'} t(\xi)\left(\left(\mathrm{sym}(\bar{\xi}\otimes I^{(2)}\otimes\bar{\xi}) - \frac{1}{2(1-\nu_0)}(\bar{\xi}\otimes\bar{\xi}\otimes\bar{\xi}\otimes\bar{\xi})\right)\right)\Big]^{-1}, \quad (38)$$

where $\tilde{\mu}_0(s)$, $\tilde{\lambda}_0(s)$, μ_1, and λ_1 are the Laplace transform of the Lame' constants of the matrix and the Lame' constants of the inclusion, respectively. Then, defining the following series S_l (with $l = 1$ to 9) as

$$S_1 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_1^2,\ S_2 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_2^2,\ S_3 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_3^2$$

$$S_4 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_1^4,\ S_5 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_2^4,\ S_6 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_3^4$$

$$S_7 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_2^2\bar{\xi}_3^2,\ S_8 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_1^2\bar{\xi}_3^2,$$

$$S_9 = \sum_{\xi}^{\pm\infty'} t(\xi)\bar{\xi}_1^2\bar{\xi}_2^2, \quad (39)$$

the final expressions of the nonzero components of the tensor $L^*(s)$ can be written in the following way:

$$s\tilde{L}^*_{11}(s) = \hat{\lambda}_0 + 2\hat{\mu}L_0 - v_f\left(\frac{S_3S_2}{\hat{\mu}_0^2} - \frac{S_5S_3 + S_6S_2}{\hat{\mu}_0^2 g} - \frac{a(S_2+S_3)}{2\hat{\mu}_0 c} + \frac{S_6S_5 - S_7^2}{\hat{\mu}_0^2 g^2} + \frac{a(S_5+S_6) + 2bS_7}{2\hat{\mu}_0 gc} + \frac{a^2 - b^2}{4c^2}\right)/D$$

$$s\tilde{L}^*_{12}(s) = \hat{\lambda}_0 + v_f\left(\left(-\frac{S_9}{\hat{\mu}_0^2 g} + \frac{b}{2c\hat{\mu}_0}\right)S_3 + \frac{S_9S_6 - S_8S_7}{\hat{\mu}_0^2 g^2} - \frac{b(S_6 - S_7) - bS_8 - aS_9}{2c\hat{\mu}_0 g} - \frac{ba + b_2}{4c^2}\right)/D$$

$$s\tilde{L}^*_{13}(s) = \hat{\lambda}_0 - v_f\left(\left(-\frac{S_8}{\hat{\mu}_0^2 g} + \frac{b}{2c\hat{\mu}_0}\right)S_2 - \frac{S_8S_5 - S_9S_7}{\hat{\mu}_0^2 g^2} + \frac{b(S_5 - S_7) - aS_8 - bS_9}{2c\hat{\mu}_0 g} + \frac{ab + b_2}{4c^2}\right)/D$$

$$s\tilde{L}^*_{22}(s) = \hat{\lambda}_0 + 2\hat{\mu}L_0 - v_f\left(\frac{S_3S_1}{\hat{\mu}_0^2} - \frac{S_4S_3 + S_6S_1}{\hat{\mu}_0^2 g} - \frac{a(S_1+S_3)}{2\hat{\mu}_0 c} + \frac{S_6S_4 - S_8^2}{\hat{\mu}_0^2 g^2} + \frac{a(S_4+S_6) + 2bS_8}{2\hat{\mu}_0 gc} + \frac{a^2 - b^2}{4c^2}\right)/D$$

$$s\tilde{L}^*_{33}(s) = \hat{\lambda}_0 + 2\hat{\mu}_0 - v_f\left(\frac{S_2S_1}{\hat{\mu}_0^2} - \frac{S_4S_2 + S_5S_1}{\hat{\mu}_0^2 g} - \frac{a(S_1+S_2)}{2\hat{\mu}_0 c} + \frac{S_5S_4 - S_9^2}{\hat{\mu}_0^2 g^2} + \frac{a(S_5+S_4) + 2bS_9}{2\hat{\mu}_0 gc} + \frac{a^2 - b^2}{4c^2}\right)/D$$

$$s\tilde{L}^*_{23}(s) = \hat{\lambda}_0 + v_f\left(\left(-\frac{S_7}{\hat{\mu}_0^2 g} + \frac{b}{2c\hat{\mu}_0}\right)S_1 + \frac{S_7S_4 - S_9S_8}{\hat{\mu}_0^2 g^2} - \frac{b(S_4 - S_8 - S_9) - aS_7}{2c\hat{\mu}_0 g} - \frac{ab + b^2}{4c^2}\right)/D$$

$$S\tilde{L}^*_{44}(s) = \hat{\mu}_0 - v_f\left(-\frac{S_2}{\hat{\mu}_0} - \frac{S_3}{\hat{\mu}_0} + (\hat{\mu}_0 - \mu_1)^{-1} + \frac{4S_7}{\hat{\mu}_0(2 - 2\nu_0)}\right)^{-1}$$

$$S\tilde{L}^*_{55}(s) = \hat{\mu}_0 - v_f\left(-\frac{S_1}{\hat{\mu}_0} - \frac{S_3}{\hat{\mu}_0} + (\hat{\mu}_0 - \mu_1)^{-1} + \frac{4S_8}{\hat{\mu}_0(2 - 2\nu_0)}\right)^{-1}$$

$$S\tilde{L}^*_{66}(s) = \hat{\mu}_0 - v_f\left(-\frac{S_1}{\hat{\mu}_0} - \frac{S_2}{\hat{\mu}_0} + (\hat{\mu}_0 - \mu_1)^{-1} + \frac{4S_9}{\hat{\mu}_0(2 - 2\nu_0)}\right)^{-1}, \quad (40)$$

where

$$D = -\frac{S_3S_2S_1}{\hat{\mu}_0^3} + \frac{(S_6S_2 + S_6S_2 + S_6S_2)S_1}{\hat{\mu}_0^3 g}$$

$$+ \frac{a(S_1S_2 + (S_1 + S_2)S_3)}{2\hat{\mu}_0^2 c} + \frac{(S_5S_4 - S_7^2)S_1 + (S_6S_4 + S_8^2)S_2 + (S_5S_4 + S_9^2)S_3}{\hat{\mu}_0^3 g^2}$$

$$- \frac{(aS_5 + aS_6 + 2bS_7^2)S_1 + (aS_4 + aS_6 + 2bS_8^2)S_2 + (aS_4 + aS_5 + 2bS_9^2)S_3}{2\hat{\mu}_0^2 gc}$$

$$+ \frac{(b^2 - a^2)}{4\hat{\mu}_0 c^2}(S_1 + S_2 + S_3) + \frac{(S_5S_6 - S_7^2)S_4 - S_8^2S_5 - S_9^2S_6 - 2S_8S_9S_7}{\hat{\mu}_0^3 g^3}$$

$$+ \frac{(aS_5 + aS_6 + 2bS_7)S_4 - (aS_7 + 2bS_8 + 2bS_9)S_7 + (2bS_5 - aS_8 + 2bS_9)S_8}{2\hat{\mu}_0^2 g^2 c}$$

$$+ \frac{-aS_9^2 + (2bS_9 + aS_5)S_6}{2\hat{\mu}_0^2 g^2 c} + \frac{a(aS_4 + aS_5 + aS_6 + 2(bS_7 + bS_8 + bS_9))}{4\hat{\mu}_0 gc^2}$$

$$+ \frac{d(2(S_7 + S_8 + S_9) - (S_4 + S_5 + S_6))}{4} + \frac{a^3 - 3ab^2 - 2b^3}{8c^3}, \quad (41)$$

and

$$\begin{aligned}
a &= \mu_1 - \hat{\mu}_0 - 2\mu_1\nu_0 + 2\hat{\nu}_0\nu_1 \\
b &= -\hat{\mu}_0\nu_0 + \mu_1\nu_1 + 2\hat{\mu}_0\nu_0\nu_1 - 2\mu_1\nu_0\nu_1 \\
c &= (\hat{\mu}_0 - \mu_1)(-\hat{\mu}_0 + \mu_1 - \hat{\mu}_0\nu_0 - 2\mu_1\nu_0 \\
&\quad + 2\hat{\mu}_0\nu_1 + \mu_1\nu_1 + 2\hat{\mu}_0\nu_0\nu_1 - 2\mu_1\nu_0\nu_1) \\
d &= b^2/(\hat{\mu}_0 gc^2) \\
g &= (2 - 2\nu_0)
\end{aligned} \quad (42)$$

where $\hat{\mu}_0 = s\tilde{\mu}_0(s)$, $\hat{\lambda}_0 = s\tilde{\lambda}(s)$ and the series S_l are given by Nemat-Nasser et al. (1982) and Iwakuma and Nemat-Nasser (1983) for several geometries of the inclusions.

Undirectional Composite

For composite material reinforced by long circular cylindrical fibers, five series are different from zero and only three are independent (Nemet-Nasser et al., 1982). If the fibers are aligned with the x_1, then

$$\begin{aligned}
S_1 &= S_4 = S_8 = S_9 = 0 \\
S_2 &= S_3, S_5 = S_6.
\end{aligned} \quad (43)$$

Therefore, Eqs. (40) to (42) became

$$s\tilde{L}^*_{11}(s) = \hat{\lambda}_0 + 2\hat{\mu}L_0 - v_f\left[\frac{S_3^2}{\hat{\mu}_0^2} - \frac{S_6S_3}{\hat{\mu}_0^2 g} - \frac{aS_3}{\hat{\mu}_0 c} + \frac{S_6^2 - S_7^2}{\hat{\mu}_0^2 g^2} + \frac{aS_6 + bS_7}{\hat{\mu}_0 gc} + \frac{a^2 - b^2}{4c^2}\right]/D$$

$$s\tilde{L}^*_{12}(s) = \hat{\lambda}_0 + v_f b\left[\frac{S_3}{2c\hat{\mu}_0} - \frac{S_6 - S_7}{2c\hat{\mu}_0 g} - \frac{a + b}{4c^2}\right]/D$$

$$s\tilde{L}^*_{23}(s) = \tilde{\lambda}_0 + v_f\left[\frac{aS_7}{2\hat{\mu}_0 gc} + \frac{ba + b^2}{4c^2}\right]/D$$

$$s\tilde{L}^*_{22}(s) = \hat{\lambda}_0 + 2\hat{\mu}_0 - v_f\left[-\frac{aS_3}{2\hat{\mu}_0 c} + \frac{aS_6}{2\hat{\mu}_0 gc} + \frac{a^2 - b^2}{4c^2}\right]/D$$

$$s\tilde{L}^*_{44}(s) = \hat{\mu}_0 - v_f\left[\frac{2S_3}{\hat{\mu}_0^2} + (\hat{\mu}_0 - \mu_1)^{-1} + \frac{4S_7}{\hat{\mu}_0(2 - 2\nu_0)}\right]^{-1}$$

$$s\tilde{L}^*_{66}(s) = \hat{\mu}_0 - v_f\left[-\frac{S_3}{\hat{\mu}_0} + (\hat{\mu}_0 - \mu_1)^{-1}\right]^{-1}, \quad (44)$$

where

$$D = \frac{aS_3^2}{2\hat{\mu}_0^2 c} - \frac{aS_6S_3}{\hat{\mu}_0^2 gc} + \frac{a(S_6^2 - S_7^2)}{2\hat{\mu}_0^2 g^2 c} + \frac{S_3(b^2 - a^2)}{2\hat{\mu}_0 c^2}$$

$$+ \frac{S_6(a^2 - b^2) + S_7(ab + b^2)}{2\hat{\mu}_0 gc^2} + \frac{(a^3 - 2b^3 - 3ab^2)}{8c^3}, \quad (45)$$

and

$$\begin{aligned}
a &= \mu_1 - \hat{\mu}_0 - 2\mu_1\nu_0 + 2\hat{\mu}_0\nu_1 \\
b &= -\hat{\mu}_0\nu_0 + \mu_1\nu_1 + 2\hat{\mu}_0\nu_0\nu_1 - 2\mu_1\nu_0\nu_1 \\
c &= (\hat{\mu}_0 - \mu_1)(-\hat{\mu}_0 + \mu_1 - \hat{\mu}_0\nu_0 - 2\mu_1\nu_0 + 2\hat{\mu}_0\nu_1 \\
&\quad + \mu_1\nu_1 + 2\hat{\mu}_0\nu_0\nu_1 - 2\mu_1\nu_0\nu_1) \\
g &= (2 - 2\nu_0).
\end{aligned} \quad (46)$$

The series S_3, S_6, S_7 are given by Nemat-Nasser et al. (1982) for several values of the volume fraction of the inclusions. However, the data can be fitted with the following parabolic expressions using a least-square method (Luciano and Barbero 1994):

$$\begin{aligned}
S_3 &= 0.49247 - 0.47603v_f - 0.02748v_f^2 \\
S_6 &= 0.36844 - 0.14944v_f - 0.27152v_f^2 \\
S_7 &= 0.12346 - 0.32035v_f + 0.23517v_f^2.
\end{aligned} \quad (47)$$

Relaxation Tensor in the Time Domain

The viscoelastic behavior of the matrix material is obtained from creep or relaxation tests. A creep test provides the strain as a function of time $\epsilon(t)$ for a fixed stress level. The matrix is said to be linearly viscoelastic if the creep

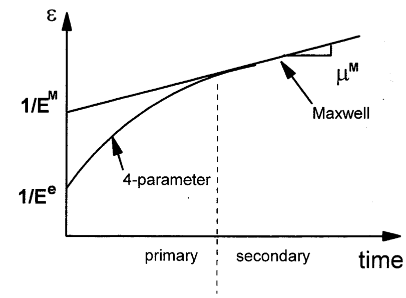

Fig. 2 Representation of creep data

compliance $M(t)$ is independent of the stress level. In this case, it is possible to write:

$$\epsilon(t) = M(t)\sigma.$$

A relaxation tests provides the stress as a function of time $\sigma(t)$ for constant applied strain, as

$$\sigma(t) = L(t)\epsilon.$$

High-temperature secondary (steady-state) creep data of metals (used in metal matrix composites) are commonly approximated by the Maxwell model (Flugge, 1967). The creep compliance of the Maxwell model, which is a series spring dash-pot system, is

$$M(t) = \frac{1}{E^M} + \frac{t}{\mu^M},$$

where μ^M is the slope of the secondary creep data (Fig. 2) and E^M represents both the elastic modulus E^e and the effect of all primary creep deformations lumped at time $t = 0$. The four-parameter model is used when a better representation of the primary creep data is desired. The four-parameter model has been used also by several authors (Skundra and Auzukalns 1973, Yancey 1990) to represent the viscoelastic behavior of polymer matrices. The model is obtained by adding a Kelvin model (also called Voigt model, which is a parallel spring dash-pot system (Flungge, 1967)) in series to the Maxwell model. The resulting creep compliance is

$$M(t) = \frac{1}{E^e} + \frac{t}{\mu^M} + \frac{1}{E^V}\left(1 - \exp\left(-\frac{tE^V}{\mu^V}\right)\right), \quad (48)$$

where $E^e \neq E^M$ (see Fig. 2). The effective relaxation modulus $\hat{E}$ is obtained from the creep compliance using the following relationship:

$$s\tilde{L}(s)s\tilde{M}(s) = \tilde{L}\tilde{M} = 1.$$

Then, from Eq. (48), the effective relaxation modulus is obtained as

$$\hat{E}_0 = \frac{E^e\eta^M(E^V + \eta^V s)s}{E^eE^V + \left(E^V\eta^M + E^e(\eta^V + \eta^M)\right)s + \eta^V\eta^M s^2}. \quad (49)$$

The Lame' properties are obtained from Eq. (49) as

$$\hat{\lambda}_0 = \frac{\hat{E}_0\nu_0}{(1 + \nu_0)(1 - 2\nu_0)} \quad (50)$$

$$\hat{\mu}_0 = \frac{\hat{E}_0}{2(1 + \nu_0)}. \quad (51)$$

Introducing these properties into Eq. (45), the coefficients of the relaxation tensor are obtained as rational functions of the Laplace variable s. The order of the polynomial in the denominator is larger than the numerator's for all the coefficients in the relaxation tensor.

After substitution of the four parameters in Eq. (49) by numerical values, the expressions of the coefficients can be easily back-transformed analytically into the time domain by standard techniques (Ogatha, 1987). Therefore, each of the coefficients of the relaxation tensor in the time domain is given by a finite sum of exponential terms with real coefficients and real-time constants.

Transversely Isotropic Material

Because of the particular geometry of the microstructure (a square array of cylinders, see Fig. 1) used to obtain Eq. (47), the relaxation tensor $L^*(t)$ for unidirectional composite represents an orthotropic material with square symmetry. In the case considered in the previous section, the directions x_2 and x_3 are equivalent and the relaxation tensor is unchanged by a rotation about x_1 of $n\pi/2$ ($n = 0, \pm 1, \pm 2\ldots$). This implies that only six components are required to describe completely the tensor.

In order to obtain a transversely isotropic relaxation tensor $C^*(t)$, equivalent in average sense to the relaxation tensor with square symmetry, the averaging procedure proposed by Aboudi (1991) is used. Then, the following expressions are obtained explicitly in terms of the coefficients of the tensor $L^*(t)$ described in the previous section:

$$C^*_{11}(t) = L^*_{11}(t)$$
$$C^*_{12}(t) = L^*_{12}(t)$$
$$C^*_{22}(t) = \frac{3}{4}L^*_{22}(t) + \frac{1}{4}L^*_{23}(t) + \frac{1}{2}L^*_{66}(t)$$
$$C^*_{23}(t) = \frac{1}{4}L^*_{22}(t) + \frac{3}{4}L^*_{23}(t) - \frac{1}{2}L^*_{66}(t)$$
$$C^*_{44}(t) = L^*_{44}(t)$$
$$C^*_{66}(t) = \frac{1}{2}(L^*_{22}(t) - L^*_{23}(t)). \quad (52)$$

This transformation can be applied also in the Laplace domain.

Comparisons With Experimental Results

Comparisons with experimental results are presented in this section. Skudra and Auzukalns (1973) measured the creep response $\epsilon(t) = \epsilon_{11}(t)$ of a glass fiber-reinforced composite with a fiber concentration $v_f = 0.54$ at three levels of tensile stress ($\sigma = \sigma_{11} = 529$ MPa, 441 Mpa and 337 Mpa). They represented the viscoelastic behavior of the ED-6 resin with the four-parameter model, using the following set of material constants: $E^e = 3.27$ GPa, $\eta^M = 8000$ GPa*hr, $E^V = 1.8$ GPa, $\eta^V = 300$ GPa*hr and $\nu_0 = 0.38$. On the other hand, the elastic properties of the glass fibers are $\nu_1 = 0.21$ and $E = 68.67$ GPa.

The analytical expressions in the time domain of the coefficients of $C^*(t)$ are obtained back transforming analytically Eqs. (44) after substituting Eq. (50) and (51),

$$\begin{aligned} C^*_{11}(t) = 37.081 &- 0.000000000378e^{-0.0186t} \\ &+ 0.00000000324e^{-0.01765t} + 1.790e^{-0.01548t} \\ &- 0.0000000001973e^{-0.00014875684943196t} \\ &+ 0.00000000164e^{-0.000144t} + 1.1068e^{-0.00013470t} \\ &- 0.6017e^{-0.0086t}\sinh(0.00851t) \\ &+ 1.929e^{-0.008654t}\cosh(0.008511t) \end{aligned}$$

$$
\begin{aligned}
C_{22}^{*}(t) = {} & 1.232e^{-0.015t} + 0.803e^{-0.0001320t} - 9.937 \\
& \times 10^{-11}e^{-0.018630t} - 4.59 \times 10^{-11}e^{-0.001487t} \\
& -0.356e^{-0.00831t}\sinh(0.00817t) \\
& +1.2594e^{-0.00831t}\cosh(0.008172t) \\
& +3.737e^{-0.0001347t} - 6.211 \times 10^{-11}e^{0.0001487t} \\
& +0.000000001749e^{-0.000144t} \\
& +0.000000003044e^{-0.01765t} \\
& +6.045e^{-0.01548t} - 0.317e^{-0.00865t}\sinh(0.008511t) \\
& +1.0177e^{-0.00865t}\cosh(0.008511t)
\end{aligned}
$$

$$
\begin{aligned}
C_{12}^{*}(t) = {} & 0.0000000002623e^{-0.01863t} + 0.00000000318 \\
& e^{-0.01765t} + 3.289e^{-0.01548t} \\
& +0.0000000001133096783e^{-0.0001485t} \\
& +0.0000000001633e^{-0.00014495t} \\
& +2.033826619e^{-0.0001347t} - 0.1745e^{-0.00865t}\sinh(0.00851t) \\
& +0.5595e^{-0.008654t}\cosh(0.008511t)
\end{aligned}
$$

$$
\begin{aligned}
C_{23}^{*}(t) = {} & -1.232e^{-0.015t} - 0.803e^{-0.000132t} + 9.93 \\
& \times 10^{-11}e^{-0.01863t} + 4.599 \times 10^{-11}e^{0.0001487t} \\
& +0.356e^{-0.00831t}\sinh(0.00817t) \\
& -1.259e^{-0.00831t}\cosh(0.00817t) \\
& +3.737e^{-0.000134t} - 6.211 \times 10^{-11}e^{-0.0001487t} \\
& +0.00000000174e^{-0.000144t} + 0.000000003044e^{-0.0176t} \\
& +6.0450e^{-0.01548t} - 0.1418e^{-10.00865t}\sinh(0.00851t) \\
& +0.454e^{-0.0086t}\cosh(0.00851t)
\end{aligned}
$$

$$
\begin{aligned}
C_{44}^{*}(t) = {} & 1.232e^{-0.015t} + 0.803e^{-0.000132} - 9.937 \\
& \times 10^{-11}e^{-0.0186t} - 4.599 \times 10^{-11}e^{-0.000148t} \\
& -0.356e^{-0.00831t}\sinh(0.00817t) \\
& +1.259e^{-0.00831t}\cosh(0.00817t) \\
& -0.0877e^{-0.00865t}\sinh(0.00851t) \\
& +0.2815e^{-0.00865t}\cosh(0.00851t)
\end{aligned}
$$

$$
\begin{aligned}
C_{66}^{*}(t) = {} & -0.1112e^{-0.00865t}\sinh(0.00851t) \\
& +0.3566e^{-0.00865t}\cosh(0.00851t) \\
& -0.7964e^{-0.00795t}\sinh(0.00781t) \\
& +3.189e^{-0.00795t}\cosh(0.00781t).
\end{aligned}
$$

A typical plot of a coefficient of the equivalent transversely isotropic relaxation tensor $C^{*}(t)$ is shown in Fig. 3 for several values of the fiber volume fraction. Comparisons of the predicted strain with the experimental data from Skudra and Auzukalns (1973) are shown in Fig. 4.

Conclusions

Analytical expressions for the Laplace transform of the relaxation tensor of composite material with general type of elastic inclusions or voids with periodic microstructure and linear viscoelastic matrix are presented. The Laplace transforms of the relaxation moduli are inverted analytically to the time domain for the case of long fiber-reinforced composites and when a four-parameter model is used to represent the viscoelastic behavior of the matrix. It is worth to noting that good agreement with available experimental data is obtained.

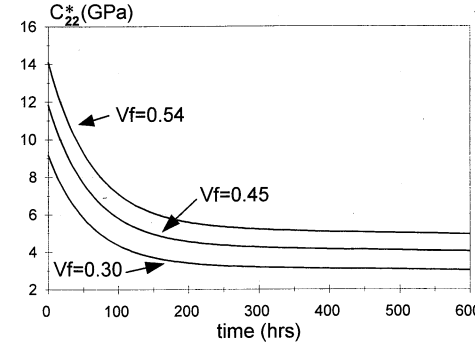

Fig. 3 Coefficient $C_{22}^{*}(t)$ of the relaxation tensor

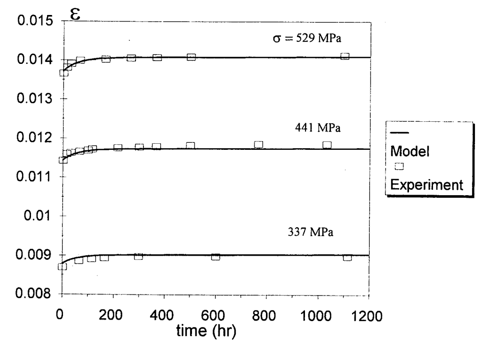

Fig. 4 Comparison with experimental results of axial creep response

The interaction effects between the constituents and the geometry of the inclusions are fully accounted.

Acknowledgments

This work was supported by the Italian National Council of Research (CNR) and the Constructed Facilities Center (CFC) at West Virginia University.

References

Aboudi, J., 1991, *Mechanics of Composite Materials*, Elsevier Science Publishers, The Netherlands.

Barbero, E. J., and Luciano, R., 1995, "Micromechanical Formulas for the Relaxation Tensor of Linear Viscoelastic Composites with Transversely Isotropic Fibers," *I. J. Solids and Structures*, Vol.. 32, No. 13, pp. 1859–1872.

Christensen, R. M., 1969, "Viscoelastic Properties of Heterogeneous Media," *J. Mech. Phys. Solids*, Vol. 17, pp. 23–41.

Christensen, R. M., 1979, *Mechanics of Composite Materials*, John Wiley and Sons, New York, p. 288.

Christensen, R. M., 1990, "A Critical Evaluation for a Class of Micromechanics Models," *J. Mech. Phys. Solids*, Vol. 38, No. 3, pp. 379–404.

Flugge, W., 1967, *Viscoelasticity*, Blaisdell Publishing, Waltham, MA.

Gurtin, M. E., and Sternberg, E., 1962, "On the Linear Theory of Viscoelasticity," *Archives for Rational Mechanics and Analysis*, Vol. 11, p. 291.

Hashin, Z., 1965, "Viscoelastic Behavior of Heterogeneous Media," ASME Journal of Applied Mechanics, Vol. 29, Vol. 32, pp. 630–636.

Hashin, Z., 1966, "Viscoelastic Fiber Reinforced Materials," *AIAA Journal*, Vol. 4, pp. 1411–1417.

Iwakuma, T., and Nemat-Nasser, S., 1983, "Composites with Periodic Microstructure," *Comput. and Struct.*, Vol. 16, No. 1-4, pp. 13–19.

Laws, N., and McLaughlin, J. R., 1978, "Self-Consistent Estimates for the Viscoelastic Creep Compliances of Composites Materials," *Proceedings of the Royal Society, London*, Vol. 39, pp. 627–649.

Luciano, R., and Barbero, J. E., 1994, "Formulas for the Stiffness of Composites with Periodic Microstructure," *Int. J. Solids and Structures*," Vol. 31, No. 21, pp. 2933–2944.

Mori, T., and Tanaka, K., 1973, "Average Stress in Matrix and Average Elastic Energy of Materials with Misfitting Inclusions," *Acta Metall.*, Vol. 21, pp. 571–574.

Mura, T., 1987, *Micromechanics of Defects in Solids*, 2nd ed., rev., Martinus Nijhoff, Dordrecht, The Netherlands.

Nemat-Nasser, S., and Hori, M., 1993, *Micromechanics: Overall Properties of Heterogeneous Solids*, Elsevier Science Publishers, The Netherlands.

Nemat-Nasser, S., Iwakuma, T., and Hejazi, M., 1982, "On Composites with Periodic Structure," *Mech. of Materials*, Vol. 1, pp. 239–267.

Nemat-Nasser, S., and Taya, M., 1981, "On Effective Moduli of an Elastic Body Containing Periodically Distributed Voids," *Q. Appl. Math.*, Vol. 39, pp. 43–59.

Nemat-Nasser, S., and Taya, M., 1986 "On Effective Moduli of an Elastic Body Containing Periodically Distributed Voids: Comments and Corrections," *Q. Appl. Math.*, Vol. 43, pp. 187–188.

Ogatha, K., 1987, *Discrete-Time Control System*, Prentice-Hall, Englewood Cliffs, NJ.

Skundra, A. M., and Auzukalns, Y. V., 1973, "Creep and Long-Time Strength of Unidirectional Reinforced Plastics in Compression," *Polymer Mechanics*, Vol. 6, No. 5, pp. 718–722.

Spiegel, M. R., 1959, *Vector Analysis*, (Schaum's outline series), McGraw-Hill, New York.

Wang, Y. M., and Weng, G. J., 1992, "The Influence of Inclusion Shape on the Overall Viscoelastic Behavior of Composites," ASME JOURNAL OF APPLIED MECHANICS, Vol. 59, pp. 510–518.

Yancey, R. N., and Pindera, M. J., 1990, "Micromechanical Analysis of the Creep Response of Unidirectional Composites," ASME *Journal of Engineering Materials and Technology*, Vol. 112, pp. 157–163.

Printed in U.S.A.

ERRATA: R. Luciano and E. J. Barbero, Analytical Expressions for the Relaxation Moduli of Linear Viscoelastic Composites with Periodic Microstructure, Journal of Applied Mechanics, 95-APM-10, 62(3), 786-793, (1995)

Eq. (44) should read

$$s\widetilde{L}^*_{11}(s) = \widehat{\lambda}_0 + 2\,\widehat{\mu}_0 - v_f\,[\frac{S_3^2}{\widehat{\mu}_0^2} - \frac{2S_6S_3}{\widehat{\mu}_0^2 g} - \frac{aS_3}{\widehat{\mu}_0\,c} + \frac{S_6^2 - S_7^2}{\widehat{\mu}_0^2 g^2} + \frac{aS_6 + bS_7}{\widehat{\mu}_0\,gc} + \frac{a^2 - b^2}{4\,c^2}]/D$$

$$s\widetilde{L}^*_{12}(s) = \widehat{\lambda}_0 + v_f\;b[\frac{S_3}{2c\widehat{\mu}_0} - \frac{S_6 - S_7}{2c\widehat{\mu}_0\,g} - \frac{a + b}{4\,c^2}]/D$$

$$s\widetilde{L}^*_{23}(s) = \widehat{\lambda}_0 + v_f\;[\frac{aS_7}{2\,\widehat{\mu}_0\,gc} - \frac{ba + b^2}{4\,c^2}]/D$$

$$s\widetilde{L}^*_{22}(s) = \widehat{\lambda}_0 + 2\,\widehat{\mu}_0 - v_f\;[-\frac{aS_3}{2\,\widehat{\mu}_0\,c} + \frac{aS_6}{2\,\widehat{\mu}_0\,gc} + \frac{a^2 - b^2}{4\,c^2}]/D$$

$$s\widetilde{L}^*_{44}(s) = \widehat{\mu}_0 - v_f\;[-\frac{2\,S_3}{\widehat{\mu}_0} + (\widehat{\mu}_0 - \mu_1)^{-1} + \frac{4\,S_7}{\widehat{\mu}_0(2 - 2\nu_0)}]^{-1}$$

$$s\widetilde{L}^*_{66}(s) = \widehat{\mu}_0 - v_f\;[-\frac{S_3}{\widehat{\mu}_0} + (\widehat{\mu}_0 - \mu_1)^{-1}]^{-1}, \tag{44}$$

Eq. (52) should read

$$C^*_{11}(t) = L^*_{11}(t)$$
$$C^*_{12}(t) = L^*_{12}(t)$$
$$C^*_{22}(t) = \frac{3}{4}L^*_{22}(t) + \frac{1}{4}L^*_{23}(t) + \frac{1}{2}L^*_{44}(t)$$
$$C^*_{23}(t) = \frac{1}{4}L^*_{22}(t) + \frac{3}{4}L^*_{23}(t) - \frac{1}{2}L^*_{44}(t)$$
$$C^*_{66}(t) = L^*_{66}(t)$$
$$C^*_{44}(t) = \frac{1}{2}\,(C^*_{22}(t) - C^*_{23}(t)) = \frac{1}{4}\,(L^*_{22}(t) - L^*_{23}(t) + 2L^*_{44}(t)) \tag{52}$$