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FINITE ELEMENTS FOR POST-BUCKLING ANALYSIS. I—THE *W*-FORMULATION

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Abstract—This paper presents a convenient formulation for the stability analysis of structures using the finite element method. The main assumptions are linear elasticity, a linear fundamental path, and the existence of distinct critical loads (i.e. no coupling between buckling modes occurs). The formulation developed is known as *W*-formulation, in which the energy is written in terms of a sliding set of incremental coordinates measured with respect to the fundamental path. In the presentation developed here, the only ingredients required to carry out the analysis are the strain–displacement and the constitutive matrices at the element level. The present formulation is compared with the so called *V*-formulation, in which the displacements refer to the unloaded state. It is shown that under the present assumptions of linear fundamental path, the advantages of the *V*-formulation are lost and both approaches are similar. An example of a circular plate under in-plane loading illustrates the procedures. Part II of this paper deals with the application to the post buckling analysis of plate assemblies made of composite materials.

1. INTRODUCTION

The post-buckling analysis of complex thin-walled components or structures requires knowledge from two important fields: stability theory and numerical techniques. The theory of elastic stability, as we know it today, was first developed by Koiter [1] for continuum structural systems, and was extended to discrete systems by researchers of University College London, notably between 1965 and 1985. A summary of the developments in structural stability for discrete systems may be found in the texts of Croll and Walker [2], Thompson and Hunt [3], Huseyin [4], Supple [5] and El Naschie [6]. Basically, the main ingredients of the theory include the identification of critical states, their classification according to the energy criterion, the study of the post-critical states, and the sensitivity of the critical states.

Problems of elastic stability are essentially nonlinear, and an approximate analysis is required to solve them. There are at least two ways to obtain the solution: via approximate numerical methods (such as continuation methods), or via approximate analytical methods (such as perturbation methods). The latter have the advantage that they employ derivatives of the energy functional at the critical state in order to construct the post-buckling solution, and these derivatives are also necessary to establish a classification of the critical state itself.

Most of the applications of the theory to buckling of structures have been made using analytical solutions, or else Ritz approximations. However, engineering problems often require the modeling of a rather complex geometry, and the finite element method has been the most convenient tool to achieve this in the last three decades. Thus, it is most desirable to have finite element solutions adapted to the needs of stability analysis.

It is difficult to identify which was the first work to couple stability theory and finite elements, (for example, Thompson and Lewis considered the circular plate in Ref. [6]); but the works of Gallagher [7], Ecer [8], Tong and Pian [6], Casciaro *et al.* [10], Casciaro and Aristodemo [11], Casciaro *et al.* [12] made important contributions during the 1970s. Koiter's methods were also employed to study snap-through buckling (see Ref. [13]).

The 1980s were largely dominated by continuation methods, but it is important to highlight the works of Batista and co-workers in Brazil (see Refs [14–16]). As mentioned before, attention was focused on computational aspects arising from continuation methods during the 1980s and the reader interested in those topics may refer to the works of Bushnell [17], Werner and Spence [18], Riks [19], Wagner and Wriggers [20], Kouhia and Mikkola [21] and Wriggers and Simo [22]. The use of finite strips (i.e. semi-analytical finite elements) in the general theory of elastic stability was developed by Graves Smith and Sridharan [23], Sridharan and Graves Smith [24] and Sridharan [25, 26].

The most recent work on finite elements for post-buckling analysis is reflected in the papers by Flores and Godoy [27–29] for shells of revolution, Mirasso and Godoy [30], who included unilateral constraints into the formulation, and Mirasso and Godoy [31] for pseudo-conservative structural systems. Koiter's theory has been adapted to a computer-aided environment [32] and to the evaluation of snapping problems [33].

A formulation based on the total potential energy has been found very convenient for elastic systems, and it has many advantages over other energy functionals. Furthermore, the total potential energy has also been one of the most popular principles for the development of finite element approximations.

A first possibility is to employ the energy functional V in its original form for the evaluation of both primary and secondary path; this is called the V-formulation. For limit point analysis, the text of Thompson and Hunt [3] contains this approach, while a formulation for bifurcation analysis was presented by Flores and Godoy [28]. An advantage of the V-formulation in problems with a non-linear fundamental path is that one can choose one path and follow it without having to calculate the others. Applications on the V-formulation are in Refs [27, 29].

The derivatives of the energy functional V may be simplified if the second derivatives, V_{ij}, are written in diagonal form. This is the so-called $V \rightarrow A$ transformation, in which case the equations necessary to identify the nature of the critical state become extremely simple. Such a transformation may be carried out for systems with only a few degrees of freedom, but it may be extremely expensive in systems with a large number of unknowns, since the diagonalization is achieved through the evaluation of eigenvalues and eigenvectors.

A third transformation has been employed in the literature, in which a set of incremental coordinates is defined; this is called the W-formulation. These coordinates retain the same direction as the original ones, but are measured from the fundamental path; they are known as sliding coordinates [3] and are defined at each value of the load parameter. Most of the work using finite elements in this field employs the W-formulation (for example see Refs [7, 14, 34]. The W-formulation can only take into account bifurcation points, and a different approach should be followed for limit points. Furthermore, it requires the computation of the fundamental path for load levels higher than the critical load, in order to calculate the post buckling path. But if the fundamental path is linear, the W-formulation may be a convenient choice, since the equations needed to obtain the post-critical path are simpler than the original V-formulation equations. The present work is limited to linear fundamental path, and has been developed within the context of the W-energy approach.

Perturbation techniques have not been as popular as other numerical techniques for non-linear analysis. The difficulties associated to perturbation methods in finite element analysis (see Ref. [35]) seem to be the complexity of the terms involved: for example, third and fourth order derivatives of the energy functional are three and four dimensional arrays. Thus, some convenient organization of the procedure is required to avoid dealing with such high order dimensional arrays. Casciaro et al. [12] noticed that the symmetric matrices of third and fourth order derivatives of the energy were never required as individual quantities, so that "—to compute and store them separately would be just a sort of computational masochism". Flores and Godoy [28] have also shown a way to contract these three and four dimensional arrays into matrices. Adequate for computer programming, their approach has been followed in this work. The notation commonly employed in finite elements has been preserved, so that the basic ingredients are the element matrices B_0, B_1, and C as in the text of Zienkiewicz and Taylor [36].

A second drawback is the poor accuracy that may be expected if large displacements are required in the post buckling range. Brezzi et al. [37] obtained theoretically the accuracy of a specific asymptotic algorithm. The results of Flores and Godoy [29] show that for thin shells, a second order perturbation analysis is accurate only within the order of the thickness of the shell. Thus, perturbation techniques are limited to the initial post-buckling range. Still, the information provided by such approach is valuable in at least three ways: first, to produce a classification of the critical state; second, to have a qualitative picture of the post-buckling path; and third, to obtain equilibrium states along the post buckling path from which a continuation technique may follow. This latter use is known as switching to a secondary path, and it has been developed in the work of Riks [19] and Flores and Godoy [29].

The above review shows the state of the art of the confluence between finite elements and perturbation techniques for structural stability. A formulation using the W-functional, and restricted to linear fundamental path is presented in Sections 2 and 3 of this paper, following the general outline of the work of Flores and Godoy [29] in the V-formulation. An advantage of this presentation is that it is fully based on conventional finite element notation, and uses contraction of third and fourth order derivatives of W. The resulting expressions are easy to program, and can even be employed on existing finite element codes. A simple example is presented in Section 4, to show the differences (and similarities) between the V- and W-formulation for this class of problems. Part 2 of this paper deals with the application of the present formulation to the analysis of plate assemblies (columns of arbitrary cross section, folded plates, etc.) and including the material properties of composites.

Notice that attention is here restricted to the analysis of isolated modes of buckling (distinct critical loads). The case of compound buckling modes (coincident or nearly coincident critical loads) will be considered in a separate work.

2. THE W-ENERGY FORMULATION

Basic equations

Following Thompson and Hunt [3] in the W-formulation we adopt a sliding set of incremental coordinates a, which are measured with respect to the single-valued fundamental path $Q^F(\Lambda)$. Thus,

$$q = Q^F(\Lambda) + a. \quad (1)$$

We shall next concentrate on a linear fundamental path, for which $Q^F(\Lambda) = \Lambda q^F$

$$q = \Lambda q^F + a \quad (2)$$

and q^F is the response in the fundamental path for $\Lambda = 1$.

The kinematic equations are next written in a form similar to Zienkiewicz and Taylor [36], and yield:

$$\epsilon = [B_0 + B_1(\Lambda q^F + a)](\Lambda q^F + a). \quad (3)$$

The constitutive equations are

$$\sigma = C\epsilon \quad (4)$$

where C is the elasticity matrix, and σ the stress tensor written in vector form.

Finally, the total potential energy V is

$$V = \frac{1}{2}\int \sigma^{\mathrm{T}}\epsilon \, \mathrm{d}v - \Lambda a^{\mathrm{T}} f \quad (5)$$

where the load vector f is incremented by a single load factor Λ.

Derivatives of strain

The derivatives of the strain vector form a matrix $[\epsilon_i]$. Each column of that matrix may be written as the derivative of ϵ with respect to a_i:

$$\epsilon_i = [B_0^i + B_1^i(a) + 2\Lambda B_1^i(q^F)] + B_i(\delta_i)a$$

where δ_i plays the role of a Kronecker delta; in the present formulation, as in Flores and Godoy [28], it is a vector with all components set to zero except for the ith component, which has a unit value. Then, ϵ_i results in

$$\epsilon_i = \{B_0^i + 2B_1^i(a) + 2\Lambda B_1^i(q^f)\}. \quad (6)$$

The second derivatives of ϵ^i (of each vector) is a matrix

$$\epsilon_{ij} = \frac{\partial \epsilon_i}{\partial a_j} = 2B_1^i(\delta_j). \quad (7)$$

Derivatives of the total potential energy

Substitution of eqns (3) and (4) into eqn (5) leads to the W functional, in terms of a

$$W[a, \Lambda] = (1/2)a^{\mathrm{T}} \int [B_0 + B_1(a) + 2\Lambda B_1(q^F)]^{\mathrm{T}} C \times [B_0 + B_1(a) + 2\Lambda B_1(q^F)] \, \mathrm{d}v a - \Lambda a^{\mathrm{T}} f. \quad (8)$$

The set of derivatives of eq (5) are:

$$W_i = \frac{1}{2}\int 2\epsilon_i \sigma \, \mathrm{d}v - \Lambda f \quad (9)$$

$$W_{ij} = \frac{1}{2}\int (\epsilon_{ij}\sigma + 2\epsilon_i\sigma_i + \epsilon\sigma_{ij}) \, \mathrm{d}v. \quad (10a)$$

$$W_{ijk} = \frac{1}{2}\int 2(\epsilon_{ij}\sigma_k + \epsilon_{ik}\sigma_j + \epsilon_i\sigma_{jk}) \, \mathrm{d}v. \quad (11a)$$

$$W_{ijkl} = \frac{1}{2}\int 2(\epsilon_{ij}\sigma_{kl} + \epsilon_{ik}\sigma_{jl} + \epsilon_{il}\sigma_{jk}) \, \mathrm{d}v. \quad (12a)$$

Substitution of eqns (4), (6) and (7) into eqns (9)–(12) leads to

$$\begin{aligned} W_{ij} = \int \{&[B_0^{\mathrm{T}i} C B_0^j] + 2[B_1^{\mathrm{T}i}(\delta_j)\sigma + 4\Lambda^2[B_1^{\mathrm{T}i}(q^f) \\ &\times C B_1^j(q^f)] + 4 \\ &\times [B_1^{\mathrm{T}i}(a) C B_1^j(a)] + 2[B_0^{\mathrm{T}i} C B_1^j(a) \\ &+ B_1^{\mathrm{T}i}(a) C B_0^j] + 4\Lambda[B_1^{\mathrm{T}i}(a) C B_1^j(q^f) \\ &+ B_1^{\mathrm{T}i}(q^f) C B_1^j(a)]\} \, \mathrm{d}v \end{aligned} \quad (10b)$$

$$\begin{aligned} W_{ijk} = \int \{&2B_1^{\mathrm{T}i}(\delta_j) C [B_0^k + 2B_1^k(a) + 2\lambda B_1^k(q^f)] \\ &+ 2B_1^{\mathrm{T}i}(\delta_k) C [B_0^j + 2B_1^j(a) + 2\Lambda B_1^j(q^f)] \\ &+ [B_0^i + 2B_1^i(a) + 2\Lambda B_1^i(q^f)]^{\mathrm{T}} C 2B_1^j(\delta_k)\} \, \mathrm{d}v \end{aligned} \quad (11b)$$

$$W_{ijkl} = 4\int \{B_i^{\mathrm{T}i}(\delta_j) C B_1^k(\delta_l) + B_1^{\mathrm{T}i}(\delta_l) C B_1^j(\delta_k) + B_1^{\mathrm{T}i}(\delta_k) C B_1^j(\delta_l)\} \, \mathrm{d}v. \quad (12b)$$

The above expressions for energy derivatives valid for any state (not only along the fundamental path) are cumbersome. However, they become simpler when evaluated at the fundamental path, in which case $a = 0$

$$W_{ijk}[0, \Lambda] = \int \{B_0^{Ti} C B_0^j + 2B_1^{Ti}(\delta_j)\sigma + 4\Lambda^2 B_1^{Ti}(q^F) C B_1^j(q^F)\}\, dv,$$

but since we are considering a linear fundamental path, non-linear terms in q^F should be neglected. Thus, $W_{ij}[0, \Lambda]$ reduces to

$$W_{ij}[0, \Lambda] = \int \{[B_0^{Ti} C B_0^j] + \Lambda[2B_1^{Ti}(\delta_j)\sigma_0]\}\, dv$$

where $\sigma = \Lambda\sigma_0$. The first term in W_{ij} is the linear stiffness matrix, K_0; while the second is usually called the load-geometry or initial stress matrix, K_σ

$$W_{ij}[0, \Lambda] = K_0 + \Lambda K_\sigma \quad (10c)$$

with

$$K_0 = \int B_0^{Ti} C B_0^j\, dv \quad (13)$$

$$K_\delta = 2\int B_1^{Ti}(\delta_j)\sigma\, dv. \quad (14)$$

The evaluated third derivatives of W reduce to

$$W_{ijk}[0, \Lambda] = \int \{2B_1^{Ti}(\delta_j) C[B_0^k + 2\Lambda B_1^k(q^F)] + 2B_1^{Ti}(\delta_k) C[B_0^j + 2\Lambda B_1^j(q^F)]\, dv + [B_0^i + 2\Lambda B_1^i(q^F)]^T C 2B_1^j(\delta_k)\}. \quad (11c)$$

inally,

$$W_{ijkl}[0, \Lambda] = W_{ijkl}. \quad (12c)$$

: will also need W'_{ij} in the analysis:

$$W'_{ij}[0, \Lambda] = K_\sigma.$$

3. STABILITY ANALYSIS IN TERMS OF W

ccording to the general theory of elastic stab- [3], a critical (bifurcation) state satisfies the fol- ng eigenvalue problem:

$$K_T x = [K_0 + \Lambda K_\sigma]x = 0. \quad (15)$$

e nature of the bifurcation depends on the sign oefficient, defined as $C = W_{ijk} x_i x_j x_k$. However, to compute this coefficient it is more convenient to define the following matrix:

$$D_1^{ij}(x)|^c = W_{ijk} x_k|^c$$

where $|^c$ denotes evaluation at the critical state. From eqn (11c), this matrix may be written as:

$$D_1^{ij}(x) = \int \{2B_1^{Ti}(\delta_j) C[B_0 + 2\Lambda B_1(q^F)]x + 2B_1^{Ti}(x) C[B_0^j + 2\Lambda B_1^j(q^F)] + [B_0^i + 2\Lambda B_1^i(q^F)]^T C 2B_1^j(x)\}\, dv. \quad (16)$$

Then,

$$C = x^T D_1(x) x. \quad (17)$$

If $C = 0$ we are in the presence of a symmetric bifurcation; while for $C \neq 0$ the bifurcation is asymmetric.

The load and displacements that define the post buckling path are expanded using perturbation analysis, and lead to

$$\Lambda = \Lambda^{(0)C} + \Lambda^{(1)C} s + \tfrac{1}{2}\Lambda^{(2)C} s^2 + \cdots \quad (18a)$$

$$a = a^{(0)C} + a^{(1)C} s + \tfrac{1}{2} a^{(2)C} s^2 + \cdots \quad (18b)$$

A suitable displacement component is here taken as perturbation parameter s. If a_1 (the first component of a) has a non-zero component in the critical mode, it may be chosen as a perturbation parameter; otherwise we would have to choose a different component of a. A convenient choice is to take the largest component of x as perturbation parameter.

Before we proceed with the analysis, it is convenient to define a second matrix $D_2^{ij}(x, x)$ as

$$D_2^{ij}(x, x) = V_{ijkl} x_k x_l \quad (19)$$

from eqn (12b) and (12c), eqn (19) results in:

$$D_2^{ij}(x, x) = 4\int \{B_1^{Ti}(\delta_j) C B_1(x) x + B_1^{Ti}(x) C B_1^j(x) + B_1^{Ti}(x) C B_1^j(x)\}\, dv. \quad (20)$$

Symmetric bifurcation

For $C = 0$, we have:

$$\Lambda^{(1)C} = 0 \quad (21)$$

$$a^{(1)C} = x. \quad (22)$$

The second order terms result in

$$K_T a^{(2)C} = -D_1(x) x \quad (23)$$

with $a_1^{(2)C} = 0$.

Stability of the bifurcation state depends on the coefficient $\tilde{W}_4$

$$\tilde{W}_4 = x^T D_2(x, x)x + 3x^T D_1(x)a^{(2)C}. \quad (24)$$

If $\tilde{W}_4 > 0$ the symmetric bifurcation is stable, while if $\tilde{W}_4 < 0$ it is unstable. Finally,

$$\lambda^{(2)C} = -\frac{\tilde{W}_4}{3x^T K_\sigma x}\bigg|^c. \quad (25)$$

Higher order derivatives could be evaluated easily.

Asymmetric bifurcation

For $C \neq 0$, we have

$$\Lambda^{(1)C} = -\frac{x^T D_1(x)x}{2x^T K_\sigma x}\bigg|^c \quad (26)$$

$$K_T q^{(2)C} = -(D_1(x)\cdot x + 2K_\sigma x \Lambda^{(1)C})|^c \quad (27)$$

with $q_1^{(2)C} = 0$. The rest of the equations can be derived similarly to the case of symmetric bifurcation.

4. INFLUENCE OF IMPERFECTIONS

The previous analysis concerns what is known as the "perfect" system under one load parameter Λ. But structural components used in practice usually have various forms of imperfections, such as deviations from the as-designed geometry, the position of the loads or the properties of the material. To study the influence of such imperfections, Koiter [1] introduced a new parameter ϵ into the energy, and focused his attention to the sensitivity of the maximum load with respect to this parameter ϵ. Thompson and Hunt [3] studied the effect of ϵ using perturbation techniques; and Flores and Godoy [28] have followed this approach using finite elements in the V-formulation. This section deals with the finite element implementation of imperfection sensitivity in the W-formulation.

Following Thompson and Hunt [3], the same parameter used for the perturbation analysis of the post-buckling path is here adopted for imperfection sensitivity. Thus, the eigenvalue, eigenvector, associated displacements and imperfections may be written as

$$\Lambda^M = \Lambda^C + \Lambda^{M(1)C}s + \tfrac{1}{2}\Lambda^{M(2)C}s^2 + \tfrac{1}{3!}\Lambda^{M(3)C}s^3$$

$$x^M = x + x^{M(1)C}s + \tfrac{1}{2}x^{M(2)C}s^2 + \tfrac{1}{3!}x^{M(3)C}s^3$$

$$a^M = a^{M(1)C}s + \tfrac{1}{2}a^{M(2)C}s^2 + \tfrac{1}{3!}a^{M(3)C}s^3$$

$$\epsilon^M = \epsilon^{M(1)C}s + \tfrac{1}{2}\epsilon^{M(2)C}s^2 + \tfrac{1}{3!}\epsilon^{M(3)C}s^3 \quad (28)$$

where Λ^C is the bifurcation load of the perfect system, obtained from eqn (15), and x is the eigenvector from the same equation. Notice that $s = a_1$ as in eqn (18). In the presence of an imperfection for which

$$\dot{W}_i x_i|^C \neq 0 \quad (29)$$

where $(\cdot)$ denotes derivative with respect to ϵ, the bifurcation is destroyed and leads to a non-linear path with a maximum at Λ^M, similar to a limit point. The eigenvector of this new critical point is x^M, and the associated displacement is a^M. The coefficients on the right hand side of eqn (28) may be obtained from perturbation of the two simultaneous conditions of equilibrium and stability, and the results are summarized in the following sections.

Asymmetric bifurcation

Solution of the set of perturbation equations lead to

$$\Lambda^{M(1)C} = 2\Lambda^{(1)C}$$

$$a^{M(1)C} = a^{(1)C} = x$$

$$\epsilon^{M(1)C} = 0$$

$$\epsilon^{M(2)C} = \frac{C}{\dot{W}_i x_i}$$

and the relation between Λ^M and ϵ^M results in first approximation as [3]

$$\Lambda^M = \Lambda^C + \alpha(\epsilon_M)^{1/2+}$$

where $1/2+$ denotes the positive square root,

$$\alpha = \pm\left(\frac{2}{\epsilon^{M(2)C}}\right)^{1/2+}\Lambda^{M(1)C}.$$

Symmetric bifurcation

In this case the main results are

$$\Lambda^{M(1)C} = 0, \quad \Lambda^{M(2)C} = 3\Lambda^{(2)C}$$

$$a^{M(1)C} = a^{(1)C} = x, \quad \alpha^{M(2)C} = \alpha^{(2)C}$$

$$\epsilon^{M(1)C} = \epsilon^{(1)C} = 0, \quad \epsilon^{M(3)C} = \frac{2\tilde{W}_4}{\dot{W}_i x_i}.$$

The first approximation to the imperfection sensitivity relation is here given by

$$\Lambda^M = \Lambda^C + \alpha(\epsilon^M)^{2/3}$$

where

$$\alpha = \frac{1}{2}\left(\frac{6}{\epsilon^{M(3)C}}\right)^{2/3}\Lambda^{M(2)C}.$$

It is clear from the above results that the only new term that needs to be computed in the first order approximation is $W_i x_i|^C$. The vector W_i depends on the specific imperfection considered.

5. ANALYSIS OF A CIRCULAR PLATE

An interesting (but yet simple) example to illustrate the use of the W-formulation solved by finite element-like approximations is the circular plate. This problem was solved by Thompson and Lewis [38] and Pandey [39], and will be studied in this section as a two degrees-of-freedom problem. The geometry of the structure is given in Fig. 1.

Basic equations

For a strain field defined by

$$\epsilon = \{\epsilon_r, \epsilon_\theta, \chi_r, \chi_\theta\}$$

and a displacement field given by

$$u = \{\bar{w}, \bar{u}\}$$

the kinematic relations are

$$\epsilon = L_0 u + L_1 u$$

where

$$L_0 = \begin{bmatrix} 0 & \dfrac{\partial()}{\partial r} \\ 0 & \dfrac{1}{r} \\ \dfrac{\partial^2()}{\partial r^2} & 0 \\ \dfrac{1}{r}\dfrac{\partial()}{\partial r} & 0 \end{bmatrix}; \quad L_1 = \begin{bmatrix} \dfrac{1}{2}\dfrac{\partial \bar{w}}{\partial r}\dfrac{\partial()}{\partial r} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

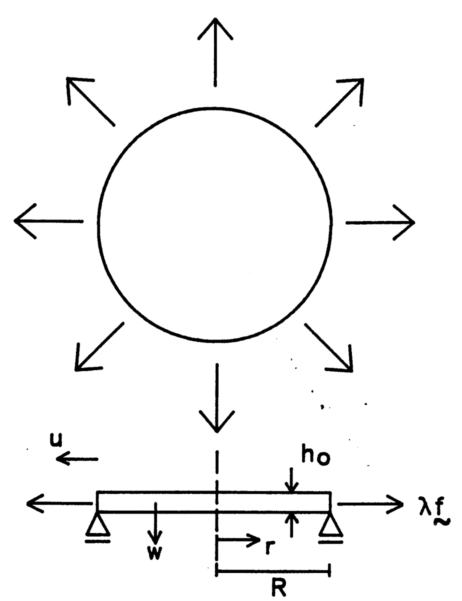

Fig. 1. Geometry of the plate considered in the example.

The constitutive equations assumed for the elastic plate are

$$\sigma = C\epsilon$$

with $\sigma = \{N_r, N_\theta, M_r, M_\theta\}$

$$C = \begin{bmatrix} C & \nu C & 0 & 0 \\ \nu C & C & 0 & 0 \\ 0 & 0 & D & \nu D \\ 0 & 0 & \nu D & D \end{bmatrix}$$

and $C = Eh/(1-\nu^2)$; $D = Eh^3/12(1-\nu^2)$.

Ritz approximation in finite element notation

To highlight the main features of the formulations developed, a simple displacement field is assumed in this section to achieve discretization of the problem. The approximation considers only the shortening of the plate and the transverse displacement deflecting in a half wave configuration. But the analysis is carried out using the present notation in order to show the main features of the W-formulation. Thus

$$u = \Phi a$$

where $u = \{a_1, a_2\}$ and Φ is the matrix of interpolation functions, given by

$$\Phi = \begin{bmatrix} \cos\dfrac{\pi r}{2R} & 0 \\ 0 & \dfrac{r}{R} \end{bmatrix}.$$

This is not the exact solution considered in Thompson and Hunt [3], but it is a good approximation to obtain a numerical approximation.

The B matrices (containing the strain–nodal displacements relations) result in

$$B_1 \equiv L_1 \Phi = \begin{bmatrix} 0 & \dfrac{1}{R} \\ 0 & \dfrac{1}{R} \\ -\left(\dfrac{\pi}{2R}\right)^2 \cos\dfrac{\pi r}{2R} & 0 \\ -\dfrac{\pi}{2Rr}\sin\dfrac{\pi r}{2R} & 0 \end{bmatrix};$$

$$B_1 \equiv L_1 \Phi = \begin{bmatrix} \dfrac{1}{2}\left(\dfrac{\pi}{2R}\sin\dfrac{\pi r}{2R}\right)^2 a_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

The linear stiffness matrix, K_0, has the explicit form

$$K_0 = \begin{bmatrix} \frac{D}{R^2}\left(\frac{\pi}{2}\right)^2 4(v + 1.191) & 0 \\ 0 & 2\pi C(1+v) \end{bmatrix}$$

and the load vector may be written as

$$f = \{0, 2\pi R\}$$

for a unit value of Λ.

Finally, the load–geometry matrix K_σ is

$$K_\sigma = \begin{bmatrix} 5.447 & N_r^f & 0 \\ 0 & & 0 \end{bmatrix}.$$

Fundamental path and critical state

The fundamental path results from the linear equilibrium condition

$$K_0 q^F - f = 0$$

leading to $q^F = \{0, R/C(1+v)\}$.

The first critical point may be obtained from the linear eigenvalue eqn (15), and yields

$$\Lambda^C = -\frac{D}{R^2}\frac{4(v + 1.191)}{1.41}$$

and

$$x = \{1, 0\}.$$

Analysis using the W-formulation

First, we study the nature of the critical state.

$$f^T \cdot x = 0$$

so that we are in the presence of a bifurcation. The $D_1(x)$ matrix is evaluated as

$$D_1(x) = \frac{C}{R}(1+v)\frac{\pi^3}{2}\left(\frac{1}{4} + \frac{1}{\pi^2}\right)\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and the matrix of contracted fourth derivatives results in

$$D_2(x, x) = \frac{C}{R^2}\frac{3\pi^5}{8}\left(\frac{3}{16} + \frac{1}{\pi^2}\right)\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

The nature of the bifurcation point is investigated by means of the coefficient

$$C = x^T \cdot D_1(x) \cdot x = 0$$

showing that it is a symmetric bifurcation. The first order perturbation solution is

$$\Lambda^{(1)C} = 0; \quad q^{(1)C} = x.$$

The second order terms in the perturbation expansion are next computed from eqn (23), leading to:

$$a^{(2)C} = \{0, -0.866/R\}$$

The stability coefficient $\tilde{W}_4$ results from eqn (24) as

$$\tilde{W}_4 = x^T \cdot D_2(x, x) \cdot x + 3x^T \cdot D_1(x) \cdot a^{(2)C}$$

$$= [10.55 - 4.50(1+v)]\frac{C}{R^2}$$

but since $v \leqslant 0.5$, then

$$\tilde{V}_4 > 0$$

and the symmetric bifurcation is stable. Next, we compute the curvature from eqn (25):

$$\Lambda^{(2)C} = -\frac{C}{6R^2}\left[(1+v)(4+\pi^2) - \pi^2\frac{(3\pi^2+16)}{4+\pi^2}\right]$$

or else

$$\Lambda^{(2)C} = (0.867v - 1.161)\frac{C}{R^2}.$$

The post buckling path may then be written as:

$$q_2 = -\frac{1}{R}\left[\frac{0.236v - 0.43}{1+v}h^2 + 0.434q_1^2\right]$$

$$\Lambda = \frac{C}{R^2}\left\{(0.236v - 0.43)h^2 + (0.867v - 1.161)\frac{q_1^2}{2} + \cdots\right\}.$$

Analysis using the V-formulation

We evalute the contracted third order derivatives $D_1(x)$ from Flores and Godoy [28], and the value is coincident with the one obtained in the W-formulation. Matrix $D_2(x, x)$ is also identical, as expected. $q^{(2)C}$ is coincident with the value of z in the V-formulation, and $\tilde{W}^4 = \tilde{V}_4$. The curvature of the path, $\Lambda^{(2)C}$ is the same as in the W approach.

The value of $q^{(2)C}$ are different, since in the V-formulation is not measured from the fundamental path, and leads to

$$q_2 = -\frac{1}{R}\left\{\frac{0.236v - 0.43}{1+v}h^2 + 0.434[1 + (1+v)(v - 1.339)]q_1^2\right\}.$$

We conclude that the more complex V-formulation does not present any advantage over the present

W-formulation if the fundamental path is linear. Furthermore, the computations lead to the same matrices of contracted third and fourth derivatives, and hence to the same results.

5. CONCLUSIONS

This work presents a W-formulation for post-buckling analysis, under the assumptions of

- linear elastic material,
- linear fundamental path,
- distinct critical modes.

If we compare the present approach with the more general V-formulation, we notice that the main advantage of the latter is that it is not necessary to calculate the fundamental path for equilibrium states beyond the first critical load, in order to obtain the secondary path. This advantage is lost under the present assumptions, for which the fundamental path is linear and no effort is required in the W approach to calculate it at post-critical levels of load.

It has been shown that both approaches can be written using the same notation and following a similar procedure of contracting three and four dimensional arrays. An example illustrates the calculations required for the solution of a problem, and it is clear that the complexity of the expressions involved is not higher than in any continuation method. In the present perturbation approach, the solution not only provides the post-buckling path, but also the nature of the critical state comes naturally from the analysis.

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