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Euler buckling of pultruted composite columns

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Pultruted composite structural members with open or closed thin-walled sections are being used as columns for structural applications where buckling is the main consideration in the design. An analytical model for Euler buckling is developed herein. Failure envelopes for some commercially available structural shapes are presented. The analytical model presented can be used to predict the behavior of any new material.

1 INTRODUCTION

Pultruted composite beams and columns are being extensively used for civil engineering structural applications. They have many advantages over conventional materials (steel, concrete, wood, etc.), such as light weight and high corrosion resistance. Mass production of composite structural members (e.g. by pultrusion) makes composite materials cost competitive with conventional ones.

In the pultrusion process, fibers are pulled through a heated die that provides the shape of the cross-section to the final product. Pultrusion is a continuous process of prismatic sections of virtually any shape.[1] Other mass production techniques like automatic tape layout can also be used to produce prismatic sections.

Pultruted structural members have open or closed thin-walled cross-sections. For long composite columns, overall (Euler) buckling is more likely to occur before any other instability failure and the buckling equation has to account for the anisotropic nature of the material. For short columns, local buckling occurs first leading either to large deflections and finally to overall buckling, or to material degradation due to large deflections (crippling). Because of the large elongation to failure allowed by both the fibers and the resin, the composite material remains linearly elastic for large deflections and strains unlike conventional materials that yield (steel) or crack (concrete) for moderate strains. Therefore, buckling is the governing failure for this type of cross-sections and the critical buckling load is directly related with the carrying capacity of the member. The Euler buckling problem is considered herein.

Buckling is an instability phenomenon that causes failure on a structure and is accompanied by large deflections and non-linear behavior. Buckling failure has been observed in long columns under axial loading where failure occurs for axial stress much lower than the yield stress of the material. Euler, based upon this fact, concluded that this instability was due to the geometry of the column (i.e. length l and bending stiffness EI) and he solved the problem mathematically.[2] Many researchers followed the same procedure in order to solve more specific cases, where the boundary conditions or eccentricities in application of the load are important.[3,4] Considering a long simply supported column under axial loading there is, from a mathematical viewpoint, an infinite number of buckling loads, each one associated with a specific deflection shape (buckling mode shape). The minimum buckling load is the critical buckling load of interest. It is assumed that once this load is reached, the structure has failed. Approximate methods of analysis, such as energy method, finite differences method, and finite element method, have been employed to solve buckling problems, whose mathematical solution may be very difficult to obtain in closed form.[5]

In the case of short columns, Euler's theory cannot be applied because short column buckling failure is associated more likely with local buckling (i.e. buckling of a part of the cross-section of the column) or material failure that can be encountered before any instability failure. A very short column of solid cross-section, for instance, is expected to fail due to material failure. On the other hand, hollow cross-section may fail due to local buckling of part of the cross-section.

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The case of local buckling of a cross-section with thin parts, such as wide flange I-beams and thin-walled box-beams, is considered by Barbero and Raftoyiannis,[6] Vakiener,[7] Yuan,[8] Raftoyiannis[9] and Barbero.[10] The problem of local buckling has been already considered for steel cross-sections and considerable research has been done in this area in order to increase the carrying capacity of a steel member against local buckling by introducing stiffeners (Desmond,[11,12] Dwight,[13] Sharp,[14] Winter,[15] Bleich[16]).

For a long composite column, the Classical Buckling Theory[4] in combination with basic concepts of the Classical Lamination Theory (CLT)[17,18] are applied in order to determine the bending stiffness of the column and the critical buckling load. The beam bending stiffness can be obtained experimentally from full size tests[19] or by considering each part of the cross-section (flange and web) as an orthotropic plate with properties determined by coupon testing.[20]

For the design of new composite beams optimized for buckling strength, it is advantageous to be able to predict member properties from the constituent (fiber, resin) properties and their arrangement in the cross-section (stacking sequence). Berkowitz developed expressions[21] for the bending, axial, and shear stiffness of laminated rectangular beams with laminae stacked perpendicularly to the plane of bending (horizontally). Bert developed a simple expression[22] for the shear correction factor, needed when using Timoshenko beam theory, that complements the work of Berkowitz. Bank developed expressions[20] for the shear correction factor of beams composed of thin orthotropic walls.

Angle-ply layers in both web and flanges are desirable to increase the local buckling strength and the efficiency of connections. Arbitrary orientation of the fibers introduces shear–extension coupling that must be accounted for in the computation of the bending stiffness. In this paper, the bending stiffness is computed directly from the description of the cross-section used by the manufacturer. In this way, an efficient optimization of the section for maximum buckling strength can be attempted.

Although pultruted beams are not manufactured by lamination, they do contain different material combinations through the thickness, thus justifying the use of lamination theory. Under this theory, each layer is modeled as a homogeneous equivalent material that macroscopically behaves similarly to the fibrous composite. Next, CLT is used to model an entire flange or web as a yet equivalent homogeneous material. Flanges and web can be dealt with separately for ease of interpretation. The beam can be derived from the plate stiffnesses of CLT or by reformulating CLT foı beam bending.

2 BEAM STIFFNESS FROM CLT

Using CLT, the stiffness components of an anisotropic laminated plate (e.g. the flange) can be computed. Hence, the constitutive equation for each laminate is

$$\begin{Bmatrix} \{N\} \\ \{M\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\varepsilon\} \\ \{\kappa\} \end{Bmatrix} \tag{1}$$

where $\{N\}$, $\{M\}$, $\{\varepsilon\}$ and $\{\kappa\}$ are stress resultants, moment resultants, strains and curvatures respectively. The coefficients A_{ij}, B_{ij} and D_{ij} are defined for the flanges of thickness $h=t^{\mathrm{f}}$ and width b (Fig. 1) as

$$(A_{ij}, B_{ij}, D_{ij}) = \frac{1}{b}\int_{-b/2}^{b/2}\int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2)\,\mathrm{d}y\,\mathrm{d}z = \int_{-h/2}^{h/2} \bar{Q}_{ij}(1, z, z^2)\,\mathrm{d}z \tag{2}$$

and for the webs of thickness $a=t^{w}$ and depth d as

$$(A_{ij}, B_{ij}, D_{ij}) = \frac{1}{a}\int_{-a/2}^{a/2}\int_{-d/2}^{d/2} \bar{Q}_{ij}(1, 0, z^2)\,\mathrm{d}y\,\mathrm{d}z = \left(\frac{d}{a}, 0, \frac{d^3}{12a}\right)\int_{-a/2}^{a/2} \bar{Q}_{ij}\,\mathrm{d}y \tag{3}$$

where $\bar{Q}_{ij}$ are the transformed reduced stiffness.[17]

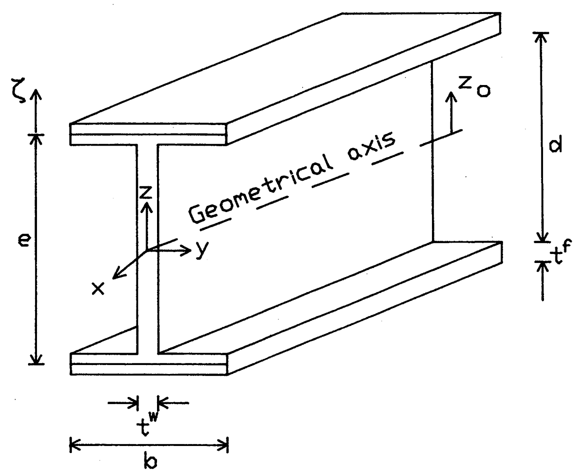

Fig. 1. Typical structural shape and coordinate system.

The units for A_{ij}, B_{ij} and D_{ij} are (kN/m), (kN) and (kNm), respectively. In using this approach, the cross-section is transformed to an equivalent laminate of width b and height h. For that reason, the webs are spread to the width of the flanges by introducing a width correction factor a/b.

Applying the parallel axis theorem[18] with respect to the middle surface of the cross-section, a constitutive matrix for the whole cross-section can be determined. For a symmetric I-beam of width b and depth $d+2h$ the total stiffness components are

$$A_{ij}=\frac{a}{b}A^{w}_{ij}+2A^{f}_{ij}$$

$$D_{ij}=\frac{a}{b}D^{w}_{ij}+2D^{f}_{ij}+2\left(\frac{e}{2}\right)^{2}A^{f}_{ij} \tag{4}$$

For a symmetric box-beam of width b and depth $d+2h$ the stiffness components are

$$A_{ij}=2\frac{a}{b}A^{w}_{ij}+2A^{f}_{ij}$$

$$D_{ij}=2\frac{a}{b}D^{w}_{ij}+2D^{f}_{ij}+2\left(\frac{e}{2}\right)^{2}A^{f}_{ij} \tag{5}$$

where $e=d+h$. Note that symmetry involves not only geometry symmetry but also material symmetry with respect to the middle surface.

So far, the analysis is similar to that of Vinson and Sierakowski[23] or Tsai[24] and is valid for a plane strain situation in the y-direction (Fig. 1). The reduced stiffnesses (Q_{ij}) are obtained using CLT[17] that assumes plane stress through the thickness of the flanges or web (Fig. 1). However, a plane stress assumption must be used through the width of the beam,[21] i.e.

$$N_y=N_{xy}=M_y=M_{xy}=0 \tag{6}$$

Enforcing eqn (6) on the consitutive eqn (1), the following can be written:

$$M_x^{tot}=M_x b=D\kappa_x \tag{7}$$

where

$$D=\left[D_{11}+\frac{2D_{16}D_{26}D_{12}-D_{66}D_{12}^{2}-D_{22}D_{16}^{2}}{D_{22}D_{66}-D_{26}^{2}}\right]b \tag{8}$$

Chen and Yang presented a similar expression[25] but limited to rectangular cross-sections.

3 LAMINATED BEAM THEORY

The computation of bending stiffness presented in the previous section is based on a plate analysis by CLT and subsequent reduction to beam theory. Therefore, the analysis of beams becomes complicated requiring understanding of plate theory to develop beam theory. This is contrary to the usual approacl. where beam theory is developed from fundamental concepts[21,22] evolving in a theory much simpler than the theory of plates. The objective of this section is to present such a theory for thin-walled beams with unsymmetric material and/or geometry and vertically as well as horizontally laminated parts. The resulting bending–extension coupling is replaced by indentifying the location of the neutral axis and formulating the stiffness with respect to this point instead of the middle surface.

The constitutive equations of a composite beam with thin, laminated flanges are developed under the assumption of plane stress[21] through the thickness and the width of the beam (Fig. 1), which is only an approximation for the case of laminates.[25,31]

$$\sigma_z \simeq \sigma_y \simeq \sigma_{yz} \simeq \sigma_{xy} \simeq 0 \tag{9}$$

Therefore, the material constitutive equations are

$$\begin{aligned} \sigma_x &= E_x \varepsilon_x \\ \sigma_{xz} &= G_{xz} \gamma_{xz} \end{aligned} \tag{10}$$

where E_x is the equivalent axial stiffness and G_{xz} is the equivalent shear stiffness of the material. For an isotropic material, $E_x = E$, the modulus of elasticity and $G_{xz} = G$, the shear modulus. The constitutive equations of a laminate (flange or web), so called laminate constitutive equations, are derived by integrating the expression of the stress resultants, to get

$$\begin{aligned} N_x &= A\varepsilon_x^0 + B\kappa_x \\ M_x &= B\varepsilon_x^0 + D\kappa_x \\ Q_x &= F\gamma_{xz} \end{aligned} \tag{11}$$

where A is the extensional stiffness, D is the bending stiffness, F is the shear stiffness, and B is the bending-extension coupling for unsymmetrical laminates (note that A, B, D and F are not matrices). For the flange, i.e. section of the beam with layers on a plane perpendicular to the plane of bending, we have

$$\begin{aligned} A^f &= b \sum_{k=1}^{N^f} \int_{\xi_{k-1}}^{\xi_k} E_x^k \, d\xi = b \sum_{k=1}^{N^f} E_x^k t_k \\ B^f &= b \sum_{k=1}^{N^f} \int_{\xi_{k-1}}^{\xi_k} \xi E_x^k \, d\xi = b \sum_{k=1}^{N^f} E_x^k t^k \xi_k \\ D^f &= b \sum_{k=1}^{N^f} \int_{\xi_{k-1}}^{\xi_k} \xi^2 E_x^k \, d\xi = b \sum E_x^k \left(t^k \xi_k^2 + \frac{t_k^3}{12} \right) \\ F^f &= b \sum_{k=1}^{N^f} \int_{\xi_{k=1}}^{\xi_k} G_{xz} \, d\xi = b \sum_{k=1}^{N^f} G_{xz}^k t_k \end{aligned} \tag{12}$$

where ξ is a local coordinate attached to the flange (Fig. 1), ξ is the position of the middle surface of the kth layer in the local coordinate system, b is the width of the flange, N^f is the number of layers of the flange, and G_{xz}^k is the out-of-plane shear modulus of the kth layer in the flange. For the web, i.e. section of the beam with layers on a plane parallel to the plane of bending, we have

$$A^w = d \sum_{k=1} \int_{y_{k-1}}^{y_k} E_x^k \, dy$$

$$B^w = 0$$

$$D^w = \frac{d^3}{12} \sum_{k=1}^{N^w} \int_{y_{k-1}}^{y_k} E_x^k \, dy \tag{13}$$

$$F^w = d \sum_{k=1}^{N^w} \int_{y_{k-1}}^{y_k} G_{xz} \, dz$$

where d is the depth of the web, N^w is the number of layers of the web, and G_{xz}^k is the in-plane shear modulus of the kth layer in the web. Equation (12) was previously used by Bert,[22] and Bert and Gordaninejad.[26]

To obtain the stiffnesses of the whole section, the contribution of the flanges (eqn (12)) and the webs (eqn (13)) are combined using the parallel axis theorem[18] with respect to the axis of symmetry of the cross-section. For example consider an I-beam as in Fig. 1. The flange has width b and thickness t^f, the web has width t^w and depth d. Therefore, the total depth of the I-beam is $h = d + 2t^f$ and $e = d + t^f$ (Fig. 1). The equivalent transformed stiffnesses are

$$A = 2A^f + A^w; \qquad F = F^w \tag{14}$$

$$B = e(K^{\text{top}} - K^{\text{bot}}) + B^{\text{top}} + B^{\text{bot}} = 0$$

Note that the contribution of the flanges to the shear stiffness is omitted in eqn (14). The assumption of constant shear strain through the thickness used in Timoshenko beam theory is unrealistic for an I-beam, where the shear strain approaches zero in the flanges.[27]

The structural equivalent properties E_x and G_{xz} for each layer apply in the structural coordinate system (Fig. 1). They are obtained by rotation from the material coordinate system to the structural coordinate system (Fig. 2). The constitutive equation for each layer, assuming a state of plane stress through the thickness of the beam is given by Jones.[17] Introducing the plane stress assumption through the width of the beam,[21] $\sigma_y = \sigma_{xy} = 0$, and replacing the constitutive equation for σ_x, we obtain

$$\sigma_x = E_x \varepsilon_x$$

$$E_x = \bar{Q}_{11} + \bar{Q}_{22} \frac{\bar{Q}_{16}\bar{Q}_{26} - \bar{Q}_{12}\bar{Q}_{66}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2} + \bar{Q}_{16} \frac{\bar{Q}_{26}\bar{Q}_{12} - \bar{Q}_{22}\bar{Q}_{16}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}^2} \tag{15}$$

As an example if $\theta = 0$ then $E_x = E_1$ (the modulus of elasticity along the fiber direction). Similarly, if $\theta = \pi/2$ then $E_x = E_2$ (the modulus of elasticity in the direction perpendicular to the fibers). To compute

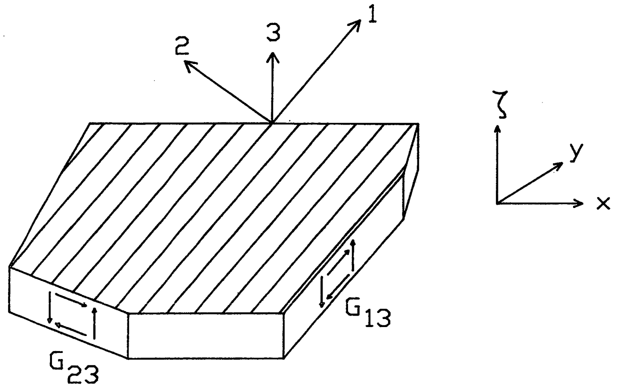

Fig. 2. Material (1-2-3) and structural (x, y, z) coordinate system for the flange.

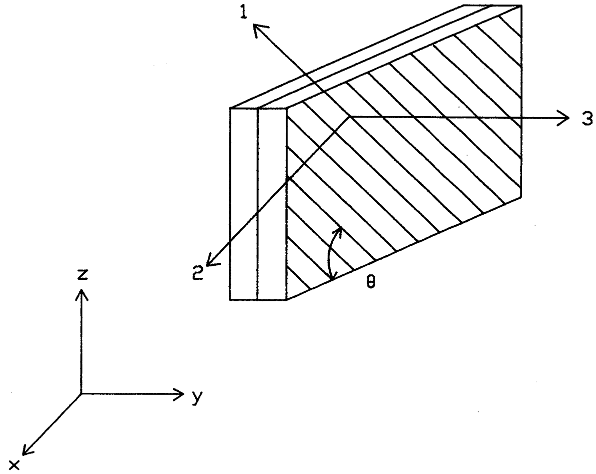

Fig. 3. Material (1-2-3) and structural (x, y, z) coordinate system for the web.

the out-of-plane shear of the flange, a typical layer laminated at an angle θ is considered. The constitutive equations for out-of-plane shear of this layer, treated as a plate under the assumption of plane stress are

$$\begin{bmatrix} \sigma_{yz} \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} \bar{C}_{44} & \bar{C}_{45} \\ \bar{C}_{45} & \bar{C}_{55} \end{bmatrix} \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} \tag{16}$$

Introducing $\sigma_{yz}=0$ into eqn (16) we obtain

$$\sigma_{xz} = G_{xz}\gamma_{xz}; \qquad G_{xz} = -\frac{\bar{C}_{45}^2}{\bar{C}_{44}} + \bar{C}_{55} \tag{17}$$

where overline indicates a rotated quantity.[28]

The lamina constitutive equations in global coordinates for a typical lamina in the web are used to compute the out-of-plane shear of the web (Fig. 3). To be consistent with beam theory, the assumption $\sigma_z=0$ must be introduced to obtain

$$\begin{bmatrix} \sigma_x \\ \sigma_{xz} \end{bmatrix} = \begin{bmatrix} \left(\bar{Q}_{11} - \dfrac{\bar{Q}_{12}^2}{\bar{Q}_{22}}\right) & \left(\bar{Q}_{16} - \dfrac{\bar{Q}_{12}\bar{Q}_{26}}{\bar{Q}_{22}}\right) \\ \left(\bar{Q}_{16} - \dfrac{\bar{Q}_{26}\bar{Q}_{12}}{\bar{Q}_{22}}\right) & \left(\bar{Q}_{66} - \dfrac{\bar{Q}_{26}^2}{\bar{Q}_{22}}\right) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \gamma_{xy} \end{bmatrix} \tag{18}$$

For the particular case of fibers oriented along the length of the beam ($\theta=0$), $Q_{16}=Q_{26}=0$ and eqn (18) reduces to the usual relationship used in beam theory:

$$\sigma_x = \left(Q_{11} - \frac{Q_{12}^2}{Q_{22}}\right)\varepsilon_x = E_x\varepsilon_x$$
$$\sigma_{xz} = Q_{66}\gamma_{xz} = G_{xz}\gamma_{xz} \tag{19}$$

If the material is isotropic $E_x=E$ and $G_{xz}=G$ in eqn (19). It is customary for unsmmetrically laminated beams to compute the location of the neutral axis rather than to use a bending–extension coupling coefficient as in CLT. The structural behavior of a composite beam depends entirely on the bending stiffness D, axial stiffness A, and shear stiffness F. The position of the neutral axis can be found by repeating the integration of eqns (12)–(13) with respect to a coordinate system located at a distance ξ_0 from the middle surface, in such a way that the bending–extension coupling vanishes,

$$B^f = b\sum E_x^k t_k(\bar{\xi}_k - \xi_0) = 0 \tag{20}$$

therefore

$$\xi_0 = \frac{\sum E_x^{\,k} t_k \bar{\xi}_k}{\sum E_x^{\,k} t_k} = \frac{B^f}{A^f} \tag{21}$$

While ξ_0 in eqn (21) is the location of the neutral axis for laminate, eqn (22) describes the position of the neutral axis for a general section of a thin-walled beam or column,

$$z_0 = \frac{B}{A} \tag{22}$$

where eqn (14) gives B and A. The axial stiffness A and shear stiffness F remain unchanged. An expression similar to eqn (22) was derived by Bert[22] for laminated beams of constant cross-section and by Bert and Gordaninejad[26] for bimodular materials based on the assumption that the in-plane normal strain vanishes at the neutral axis. The bending stiffness D with respect to the neutral axis is computed using the parallel axis theorem

$$D = D^{web} + D^{top} + D^{bot} + A^{top}(e - z_0)^2 - A^{bot}(e + z_0)^2 + A^{web} z_0 \tag{23}$$

Considering transverse isotropy on each layer,[17] four material properties per layer are needed. Using micromechanics, the material properties for each lamina (E_1, E_2, ν_{12}, G_{12}) are determined from the material properties of fiber (E_f, ν_f) and matrix (E_m, ν_m).

The fiber volume fraction V_f is obtained from the pultrusion manufacturing information. For unidirectional fibers aligned with the axis of the beam $V_f = n/(y\rho t_c)$, where n is the number of roving per unit width, y is the yield (number of meters of roving weighing 1 kg), ρ is the density in kg/m^3, and t_c is the thickness of the layer. For the continuous strand mat (OC) $V_f = W/(\rho t_c)$, where W is the weight per unit area of the OC mat.

For the material under study, $E_f = 72{\cdot}393$ GPa, $E_m = 3{\cdot}445$ GPa, $\nu_f = 0{\cdot}22$, and $\nu_m = 0{\cdot}35$. Due to the high tension that pultrusion exerts on the fibers, the fiber misalingment factor used is $K = 1$. As an example, consider an E-glass-vinylester pultruted material with $V_f = 0{\cdot}25$ for which the following are obtained $E_1 = 20{\cdot}632$ GPa, $E_2 = 4{\cdot}433$ GPa, $\nu_{12} = 0{\cdot}318$. These predictions correlate well with experimental data.[29]

The elasticity solution with contiguity is used for the determination of the shear modulus. Using eqn (3·68) in Ref. 17, $G_{12} = 1{\cdot}985$ GPa is obtained. The predicted value does not correlate well with experimental data. Therefore, the stress partition parameter[18] is obtained using experimental data for currently produced pultruted material and it is assumed to remain constant while varying the fiber volume and resin properties during material optimization studies. Predictions using this approach correlate well with experimental data.[29]

In unidirectional composites all the fibers have a specific orientation in the matrix for each particular layer. A special case is when the fibers are randomly oriented in the matrix. The composite acts as a plane-isotropic material and the properties were obtained using the following formulas[30]

$$E = \frac{3}{8}E_1 + \frac{5}{8}E_2, \qquad G = \frac{1}{8}E_1 + \frac{1}{4}E_2, \qquad \nu = \frac{E}{2G} - 1 \tag{24}$$

4 BUCKLING EQUATION AND CRITICAL LOAD

The governing equation for buckling is obtained by introducing eqn (9) into the Euler buckling equation for a pinned-pinned perfect column under axial compressive load (eqn (2.1) in Ref. 4). Then, the critical buckling load is found to be

$$P_{cr} = \frac{\pi^2}{l^2} D \tag{25}$$

Equations (8) and (25) are used to draw the long column failure envelopes presented in Figs 4–8, for structural shapes produced by Creative Pultrusions Inc. The failure envelopes are produced by using the analysis presented herein for Euler buckling and the local buckling predictions of Raftoyiannis.[9]

Five different wide-flange I-beam sections and a thin-walled box-beam section are considered. It must be noted that for the $10 \times 10 \times 0{\cdot}432$ cm (Fig. 4) and $15 \times 15 \times 0{\cdot}635$ cm (Fig. 5), the transition from local to Euler buckling occurs for relatively short lengths (1–1·5 m). Therefore, accurate prediction of Euler buckling becomes quite important for structural applications. For the

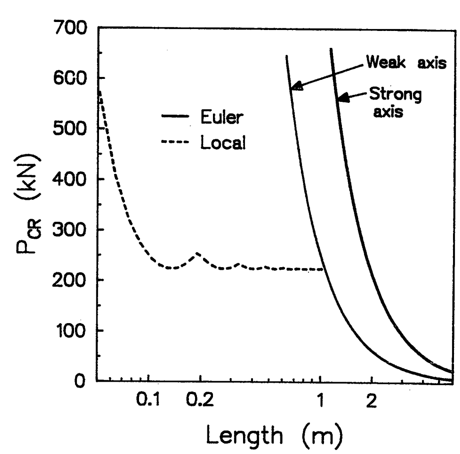

Fig. 4. Buckling failure envelope for a $10 \times 10 \times 0{\cdot}432$ cm wide flange I-beam from Creative Pultrusions Inc.

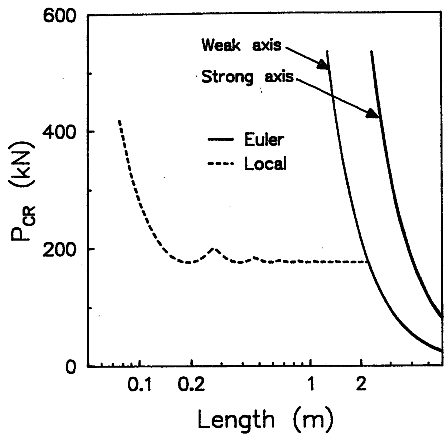

Fig. 5. Buckling failure envelope for a $15 \times 15 \times 0{\cdot}635$ cm wide flange *I*-beam from Creative Pultrusions Inc.

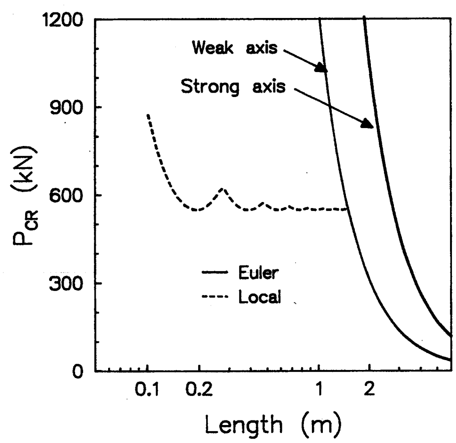

Fig. 6. Buckling failure envelope for a $15 \times 15 \times 0{\cdot}952$ cm wide flange *I*-beam from Creative Pultrusions Inc.

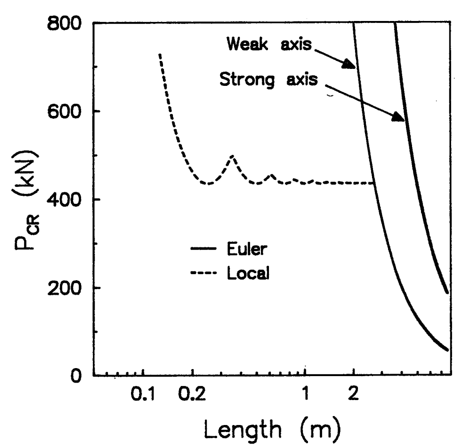

Fig. 7. Buckling failure envelope for a $20 \times 20 \times 0{\cdot}952$ cm wide flange *I*-beam from Creative Pultrusions Inc.

$15 \times 15 \times 0{\cdot}952$ cm (Fig. 6) and $20 \times 20 \times 0{\cdot}952$ cm (Fig. 7), the transition occurs for 2–3 m long columns. Therefore, the behavior of columns of intermediate length becomes critical. The interaction between local and Euler column buckling

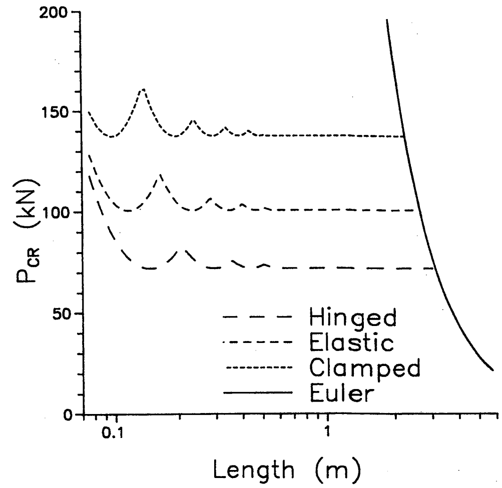

Fig. 8. Buckling failure envelope for a $10 \times 10 \times 0{\cdot}432$ cm thin-walled box-beam from Creative Pultrusions Inc.

behavior that defines the intermediate length for structural shapes is currently under investigation. The buckling envelopes for a $10 \times 10 \times 0{\cdot}432$ cm box-beam is shown in Fig. 8.

5 CONCLUSIONS

The analytical solution developed herein has been used to predict critical buckling loads as well a the buckling mode failure for commercially available pultruted columns. Instability failure is very important because it leads to the collapse of the member. Prediction of the buckling load is very important for the prediction of ultimate axial strength of pultruted columns. The analytical solutions presented are based on the description of the cross-section used by the manufacturer. A comprehensive analysis, from the properties of fiber and matrix to the determination of the buckling load has been presented, which can be used by the manufacturer to tailor the properties of the material for specific applications.

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