Predicting High Temperature Ultimate Strength of Continuous Fiber Metal Matrix Composites

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ABSTRACT: A model to predict the high temperature ultimate strength of a continuous fiber metal matrix composite (CFMMC) has been developed. The model extends the work of Rosen by including high temperature processes such as matrix creep, fiber-matrix debond, and the effects of randomly spaced fiber breaks which typically exist in the MMC prior to loading. A finite element model (FEM), developed in the form of a representative volume element (RVE), is used to calculate the time-dependent stress field surrounding a fiber break. Variables included in the calculation are process-related parameters such as the fiber diameter, the fiber-matrix interface strength, and interface roughness. Statistical analysis is used to infer the strength of a large composite sample from the stress analysis of a single break provided by the FEM.

INTRODUCTION

CONTINUOUS FIBER METAL matrix composites (CFMMCs) utilizing high-melting point matrices (like titanium) are seen as attractive material candidates in applications where high strength or stiffness-to-weight ratios at high temperature are required. They can be commercially successful only if their mechanical properties, particularly in the longitudinal direction, are superior to presently-used materials by a margin large enough to justify their expected high cost. Processing variables such as temperature and pressure during fiber-matrix consolidation, and the choice of fiber coating and fiber diameter ultimately control the mechanical properties of a composite. The goal of this article is to predict how fiber-matrix interface properties, fiber diameter, and preexisting fiber failures affect the high temperature ultimate strength of a composite.

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The longitudinal tensile strength of a composite is far more difficult to predict than such properties as E, CTE, and ν which are all well-approximated using micromechanics. Fortunately, Rosen [1] has already developed a reasonably successful model which predicts the ultimate strength of a ductile matrix, brittle elastic fiber composite at room temperature. The ultimate strength of the composite is controlled by the fiber strength distribution, the percentage of the load carried by the fibers, and the ineffective length δ (Figure 1) over which a fiber can be considered unloaded next to a fiber break.

When a constant load is applied at elevated temperatures, matrix creep redistributes load onto the fibers and the ineffective length, δ, increases over time. As δ increases, the fiber bundle strength decreases. The combination of an increasing fiber bundle stress and a decreasing bundle strength can cause composite failure at a surprisingly low constant applied stress.

Lifshitz and Rotem [2] modeled this problem analytically by including a viscoelastic matrix into Rosen's model. Their analysis predicts an increase in δ over time (albeit relatively small due to an infinitely strong fiber-matrix interface). However, rather than update the effective fiber strength based on the existing value of δ, a complex formulation is used to calculate the fiber strength using the original value of δ. When the fiber-matrix interface is weak, the time-dependent increase in δ can be much greater than predicted by their analysis due to fiber-matrix slip. In addition, it is more accurate and simple to base strength on the ac-

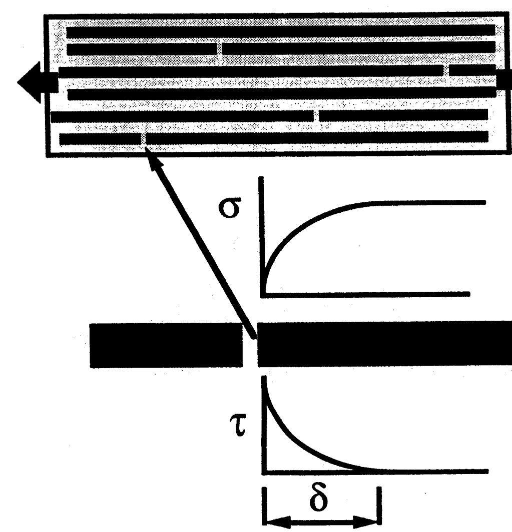

Figure 1. *Definition of ineffective length, δ.*

tual, time-dependent value of δ. Finally their model cannot account for fiber failures which exist in the composite prior to loading. In MMCs, this preexisting fiber break density is often significant.

In this article, a simple method to calculate the ultimate strength of a composite is presented. A FEM is used to calculate the time-dependent stress field surrounding a fiber break. The finite element model provides a time-dependent approximation of δ that cannot be determined analytically. The results of this model, coupled with a knowledge of the fiber break density which may exist in the composite before a load is applied, are used to calculate values such as the time-dependent factor-of-safety and time to failure at a given applied load.

STATISTICAL ANALYSIS

A "dry" fiber bundle is defined, in this article, as a number of parallel fibers of some given length and diameter which, if unbroken, carry the same load. After a fiber within a dry bundle fails, the load it carried is shared equally by the remaining unbroken fibers. A dry bundle typically refers to fibers which have not yet been combined with matrix. As tensile load is slowly applied to a dry bundle of fibers which have some distribution of strength, the weaker fibers (with large flaw sizes) begin to fail and the stress on the remaining unbroken fibers increases accordingly. The Weibull expression [3], often used to describe the cumulative probability, $F(\sigma)$, that a fiber of length L will fail at stress σ, is given as

$$F(\sigma) = 1 - \exp\left[\frac{-L}{L_0}\left(\frac{\sigma}{\sigma_0}\right)^m\right] \tag{1}$$

The values of σ_0 and m, which represent the characteristic strength of the fiber and the dispersion of fiber strength respectively, can be determined from fiber strength experiments. L_0 in Equation (1) is the characteristic length associated with stress σ_0. Equation (1) can be simplified as shown below:

$$\alpha = \frac{1}{L_0\sigma_0^m} \tag{2}$$

$$F(\sigma) = 1 - \exp[-L\alpha\sigma^m] \tag{3}$$

Quite often, fiber vendors provide the average fiber strength, σ_{av}, for a given gauge length L, rather than α. In these cases, Equation (4) can be used to calculate the value of α.

$$\alpha = \left[\frac{\Gamma\left(1 + \dfrac{1}{m}\right)}{\sigma_{av}}\right]^m \frac{1}{L_0} \tag{4}$$

The bundle stress, σ_b, is equal to the applied load divided by the total fiber

cross-sectional area. It is also equal to the product of the stress in unbroken fibers and the percentage of fibers which are unbroken as shown in Equation (5).

$$\sigma_b = [1 - F(\sigma)]\sigma \tag{5}$$

The value of σ which maximizes Equation (5), σ_m, is given by Equation (6). Catastrophic failure of the bundle occurs if the stress on the unbroken fibers exceeds this value.

$$\sigma_m = (L\alpha m)^{-1/m} \tag{6}$$

The maximum (or critical) bundle stress, σ_c, is calculated by combining Equations (5) and (6). In this article, as is often the case, the critical bundle stress is termed the bundle strength.

$$\sigma_c = (L\alpha me)^{-1/m} \tag{7}$$

The bundle strength, σ_c, decreases with increasing length, and the dependency on length increases as m decreases. The relationship between strength and length is easily explained since the probability of the existence of a flaw of any chosen size increases proportionately with length.

When a fiber breaks in a composite, the tensile load in the fiber increases from a value of zero at the break to essentially its fully loaded value (90%) over some distance δ (Figure 1). Therefore, unlike a dry bundle, a broken fiber can be considered "broken" over a relatively short distance surrounding the break. Fiber breaks more than δ apart in the axial direction are essentially decoupled. Rosen [1] showed that the effective strength of fibers in a ductile matrix, elastic fiber composite can be accurately predicted by modeling the composite as a series of dry bundles each of length δ. By substituting δ for L in Equation (7), and by invoking a number of assumptions which will be discussed, Rosen predicted that the effective strength of fibers in a composite is equal to the strength of a dry fiber bundle of length δ.

$$\sigma_c = (\delta\alpha me)^{-1/m} \tag{8}$$

Equation (9) provides a reasonable estimate of the composite ultimate strength where the fiber strain at failure is used to estimate the stress in the matrix.

$$\sigma_u = V_F\sigma_c + (1 - V_F)\sigma_m \tag{9}$$

σ_u = composite ultimate strength
σ_c = fiber bundle strength
σ_m = matrix stress
V_F = percent fiber volume

Rosen validated his model by analyzing the progression toward failure of an epoxy matrix, glass fiber composite. He found that the number of randomly located fiber breaks versus applied load correlated well with model predictions, although in general, the composite failed at stresses somewhat lower than predicted by Equation (9) [1].

Subsequent investigations have examined the magnitude of the errors introduced by Rosen's assumptions. For example, Rosen assumed that when a fiber breaks, the load carried by the fiber is shared equally by the remaining unbroken fibers in the axial plane of the break. In reality, those fibers close to the break are burdened with a relatively large load increase compared to fibers further away, making invalid the assumption that subsequent fiber failures will occur randomly. Nevertheless, Rosen noted in his experiments that fiber failures were scattered and that fiber strength variation determined the location of subsequent fiber failures much more than break-induced stress concentration sites. Experiments performed by Zweben [4] showed that locations where multiple breaks occurred did not lead to catastrophic failure until stresses "reasonably" close (approximately 90%) to Rosen's predicted value were reached. Therefore, propagating fiber failure induced by the failure of neighboring fibers was only prevalent at high stresses and did not significantly change the prediction of Rosen. He, Evans and Curtin [5] have developed a criterion to predict when Rosen's approximation of equal load sharing is valid. Qualitatively, at typical fiber volume fractions (0.3 to 0.4), their study concludes that Rosen's assumption of equal load sharing is valid if the ratio of the maximum allowable shear stress at the fiber-matrix interface to the fiber bundle stress is less than or equal to 0.1. Their study notes that the value of this ratio must be lowered as the matrix stiffness to fiber stiffness ratio becomes significantly smaller than one. It also states that the effects of local stress concentration sites are only important when composite stresses reach a significant percentage of the fiber bundle strength. The assumption of equal load sharing always produces a predicted ultimate strength that is higher than measured. Whether the discrepancy is small (10%) or large depends upon the material and geometric parameters listed above. This study will assume that equal load sharing exists.

Rosen also assumes that the composite can be approximated as a series of bundles each of length δ as defined earlier. Increasing the length of a dry bundle by a factor of two lowers the bundle strength by about 7% and 15% respectively for m values of 5 and 10 [Equation (8)]. Obviously, modeling the composite as a series of bundles of the "correct" length is required to accurately predict effective fiber strength. Since a given fiber break "communicates" with other fiber breaks within axial distance $\pm\delta$, an argument can be made that the appropriate choice of bundle length is 2δ, rather than the value of δ chosen by Rosen. Appendix 2 demonstrates that δ is an acceptable choice of bundle length—a fact which was counterintuitive to the authors.

Equation (8) also assumes that within axial length δ all broken fibers behave as though they were within a dry bundle of length δ. In reality, broken fibers within a composite carry load to varying degree within distance δ from a break. The load-carrying capability of broken fibers might be expected to increase the actual ultimate strength above the value predicted by Equation (8). Appendix 2 presents

an approach to calculate the change in the ultimate strength caused by including the load-carrying capability of broken fibers for a given assumed bundle length. The surprising result is that if a bundle length of δ is assumed, then the load-carrying capability of broken fibers has almost no effect on the bundle strength of the fibers.

Finally, Rosen assumed that each of the bundles in series has the same strength. This assumption is valid only when the number of fibers within the bundle is very large. As the number of bundles in series becomes very large and/or as the number of fibers in each bundle decreases, the dispersion of the bundle strengths increases. Obviously, the composite will fail when the weakest bundle fails. Therefore, Rosen's model always tends to overpredict the composite strength since it assumes no bundle strength dispersion exists.

In summary, Rosen's model tends to overestimate the ultimate strength by neglecting fiber failures caused by neighboring breaks, and by neglecting bundle strength dispersion. Nevertheless, it seems to accurately characterize both the failure behavior and the ultimate strength of many ductile matrix–brittle fiber composites.

An often-overlooked parameter which can significantly affect an MMC's strength is the density of fiber breaks which exist in the composite prior to the application of load. Rosen's model assumes the fiber failure density is a function only of the composite constituent properties, and the applied longitudinal load. The rest of this section describes a method which adapts Rosen's approach to incorporate the effects of fiber breaks not associated with applied load.

An initial density of fiber breaks in a composite can be equated to an initial average fiber length, L_i. These initial average lengths can be on the order of a few hundred fiber diameters [6]. Each fiber segment within the composite is unloaded, to varying degree, at both ends over length δ. If the assumption is made that the load increases linearly within the ineffective length, then the average stress within both ineffective length regions is one-half the "full" load carried by the fiber. By redistributing the load carried at each end, the fiber can be considered fully loaded over length $L_i - \delta$, and completely unloaded over length δ.

This initial average fiber length can be treated as a bundle of N fibers, each of length δ. The product of N and δ is always equal to L_i. The percentage of fibers in the bundle which are considered initially broken and unbroken, are given by Equations (10a) and (10b) respectively.

$$\%\text{ Initially Broken} = \frac{\delta}{L_i} \tag{10a}$$

$$\%\text{ Initially Unbroken} = 1 - \frac{\delta}{L_i} \tag{10b}$$

The failure probability, $F(\sigma)$, given in Equation (3), needs to be modified to account for the fact that a percentage of fibers are broken initially. Equation (11) provides an expression for $F(\sigma)$ which includes these initially-broken fibers. The subscript "t" is used to differentiate the total damage from the damage induced by applied load.

$$F_t(\sigma) = [1 - \exp(-\delta\alpha\sigma^m)]\left(1 - \frac{\delta}{L_i}\right) + \frac{\delta}{L_i} \quad (11)$$

The first term on the right-hand side represents the percentage of fibers which can be considered initially unbroken but fail at stress σ, while the second term is the percentage of fibers that can be considered initially broken. The percentage of fibers which are unbroken, $1 - F_t(\sigma)$, is given as

$$1 - F_t(\sigma) = \exp(-\delta\alpha\sigma^m)\left(1 - \frac{\delta}{L_i}\right) \quad (12)$$

The bundle stress, σ_b, is again equal to the product of the stress in unbroken fibers and the percentage of unbroken fibers.

$$\sigma_b = \sigma \exp(-\delta\alpha\sigma^m)\left(1 - \frac{\delta}{L_i}\right) \quad (13)$$

The stress in the unbroken fibers corresponding to the bundle strength, σ_c, is again easily determined by maximizing σ_b with respect to σ. The introduction of initial breaks in the composite does not change the maximum unbroken fiber stress [Equation (6) with $L = \delta$], and the bundle strength, given by Equation (14), is proportional to the value provided by Equation (8).

$$\sigma_c = \left[1 - \frac{\delta}{L_i}\right](\delta\alpha m e)^{-1/m} \quad (14)$$

Neither high temperature-related processes such as creep-induced load transfer from the matrix to the noncreeping fibers, nor the presence of preexisting fiber breaks in the composite should affect the basic framework of Rosen's model. The analytically-determined and constant value of δ used by Rosen is replaced by a time-dependent value which is calculated with a FEM. The fiber failure density in a composite with preexisting fiber breaks is always higher than is assumed by Rosen's model. However, the composite is still considered to behave as a series of dry bundles, and Equation (14) still provides the fiber bundle strength needed to estimate the ultimate composite strength. The bundle strength can be compared to the existing time-dependent fiber bundle stress to determine if or when failure will occur at a given applied stress.

NUMERICAL MODEL

A FEM using ANSYS [7] has been developed to calculate the response of a composite under longitudinal tension on a representative volume element (RVE) containing one internal fiber break. The model calculates the values of δ and the far-field fiber stress as functions of time. The far-field stress represents the fiber stress far from any breaks within the composite and is equal to the applied dry bundle stress, σ_b. The ultimate strength of the composite, governed by bundle strength, σ_c, depends upon the value of the ineffective length δ as described in the last section.

The numerical analysis will present modeling results of a titanium matrix, alumina fiber composite subjected to a constant longitudinal load at 800°F (427°C). The alumina fibers are created by a sol-gel process, and therefore, have diameters which are relatively small (20 μm). The properties of both the matrix and the fiber used in the model are listed in Appendix 1. An axisymmetric view of the finite element model is shown in Figure 2. A ring of matrix material surrounds the fiber. The thickness of the matrix ring corresponds to a composite with a 30% fiber volume. In the preliminary analysis, the fiber-matrix interface allows sliding (coulomb friction) when shear at the interface exceeds the product of the compressive force across the interface and an assumed coefficient of friction (Appendix 1). A ring of material with homogenized properties of the undamaged composite is coaxially located outside of and rigidly connected to the ring of matrix. The properties of the homogenized composite are those of the undamaged material (no fiber breaks). The undamaged homogenized composite provides the capability to more accurately calculate the stress redistribution expected to occur in the vicinity of a fiber break and makes less critical the selection of the boundary condition along the outer radius of the model. This is the typical configuration suggested by the cylindrical assemblage model [8–10]. In addition, the radius of the model, and therefore the amount of undamaged

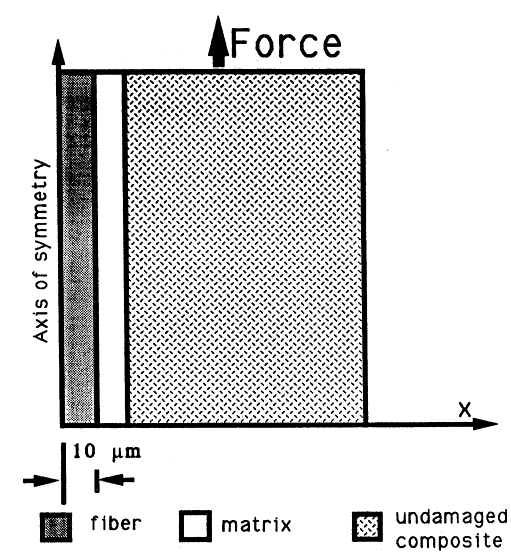

Figure 2. Model.

composite included in the unit cell, is related to instantaneous fiber break density, or average fiber length within the composite. As the RVE volume decreases, the longitudinal strain associated with a given applied load increases because the effect of the broken fiber in the center becomes proportionately larger.

The boundary conditions which are used in the model corresponding to Figure 2 are described as follows:

1. The model is axisymmetric.
2. The bottom surface of both the matrix and undamaged composite cylinders allow zero axial displacement. When the fiber is considered unbroken, its bottom surface allows no axial displacement. When the fiber is considered broken, the bottom surface of the fiber is allowed to move in the positive z direction, but not in the negative.
3. The top surface of the model remains planar in response to longitudinal loads.
4. There are no restraints to radial displacements except along the centerline of the fiber.

HOMOGENIZED PROPERTIES OF UNDAMAGED COMPOSITE

The success of modeling the stress field surrounding a fiber break hinges on accurately modeling the behavior of the undamaged composite. The composite properties E_1, E_2, CTE_1, CTE_2, G_{21}, ν_{21}, as well as creep response, are needed to predict the response of the composite to thermal and mechanical loads. These homogenized composite properties (with the exception of the creep response) were determined using micromechanics [11,12].

CREEP RESPONSE OF UNDAMAGED COMPOSITE

Analytic expressions [13] are available to estimate the creep in a continuous elastic fiber, creeping matrix composite subjected to a constant longitudinal load. A constitutive equation relating matrix stress to the creep rate governs the initial slope of the creep displacement versus time plot and a steady state creep strain is achieved after the load initially carried by the matrix is transferred to the fibers. A two-phase model, similar to the one shown in Figure 2, but without the undamaged composite, was used to calculate longitudinal creep of a continuous titanium matrix, alumina fiber composite with a 30% fiber volume fraction. Using the boundary conditions appropriate when the fiber is considered unbroken, a longitudinal stress was applied to the top surface of the model. The creep of the matrix was modeled by the equation and constants listed in Appendix 1. The longitudinal creep versus time was calculated for different applied stresses. The results are shown in Figure 3.

From Figure 3, it is clear that the final creep strain is directly proportional to applied stress, so the average strain rate over long periods of time is directly proportional to applied stress. In addition, the curves shown in Figure 3 appear to be of similar shape and differ only by a scale that is proportional to the applied

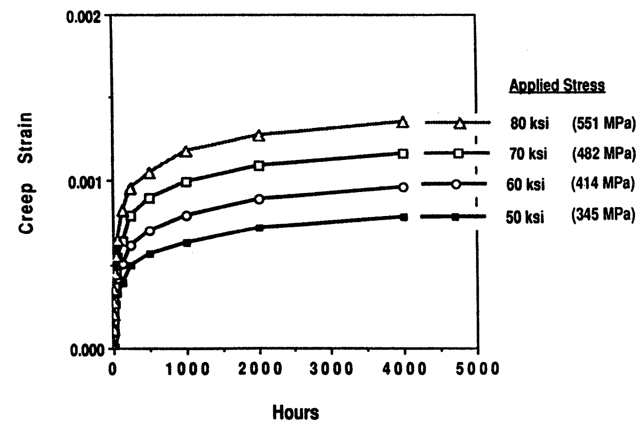

***Figure 3.** Axial creep of undamaged composite at 800°F (427°C).*

stress. Therefore, the creep strain for any constant applied stress can be well approximated by the equation below:

$$\epsilon_{cp}(t) = \epsilon_0(t)\left(\frac{\sigma_a}{\sigma_0}\right) \tag{15}$$

where

$\epsilon_{cp}(t)$ = creep strain
$\epsilon_0(t)$ = reference creep strain
σ_a = applied composite stress
σ_0 = reference applied composite stress

The longitudinal creep response of the homogenized composite to any constant applied load can be closely approximated by adapting constitutive equations provided by ANSYS. The creep strain curve corresponding to an applied load of 80 ksi (551 MPa) was chosen to represent the function $\epsilon_0(t)$.

The stress state within the undamaged composite is triaxial and its creep response is anisotropic. The creep model is only expected to accurately predict undamaged composite creep strain when it is subjected to a pure axial stress. Furthermore, the function $\epsilon_0(t)$ applies when the applied load is constant. If the stresses within a region of the undamaged composite vary significantly over the time during which load is applied, then the model may not correctly predict the local creep rate. For example, if the stress within a region of the undamaged composite increases with time, the model may underpredict the local creep rate. Nevertheless, the authors believe that the model should represent reasonably well the creep response of the undamaged composite since axial stress dominates and since the magnitude of the stress variation within the undamaged composite is relatively small.

NUMERICAL RESULTS

The following scenario was modeled: the composite modeled in Figure 2 is consolidated at 1700°F (927°C) and cooled to 800°F (427°C). After cooldown, an 80 ksi (551 MPa) longitudinal stress is applied and maintained for 4000 hours. It is assumed the fiber matrix interface strength is zero and coulomb friction provides transfer of shear stress across the fiber-matrix interface.

Following cooldown, a radial compressive stress of approximately 9.5 ksi (65.5 MPa) exists at the fiber-matrix interface. This stress, combined with an assumed coefficient of friction, μ, provides the necessary shear stress which must be overcome for slip to occur at the interface. Over the 4000 hour period of load application, the compressive stress decreases by about 50% (matrix relaxation) which causes the maximum interface shear stress to decrease accordingly.

Figures 4 and 5 show, respectively, the axial and shear stress fields which exist immediately after the 80 ksi stress is applied and 4000 hours later. The stress in the fibers near the top of the model, equal to σ_b, increases from 145 ksi (1000 MPa) to 244 ksi (1682 MPa) (Figures 4 and 5). The value of δ also increases (Figures 4 and 5). δ is obtained by determining the location where the fiber stress is 90% of the far field bundle stress.

It is convenient to normalize the value of δ by the fiber diameter D. The δ/D

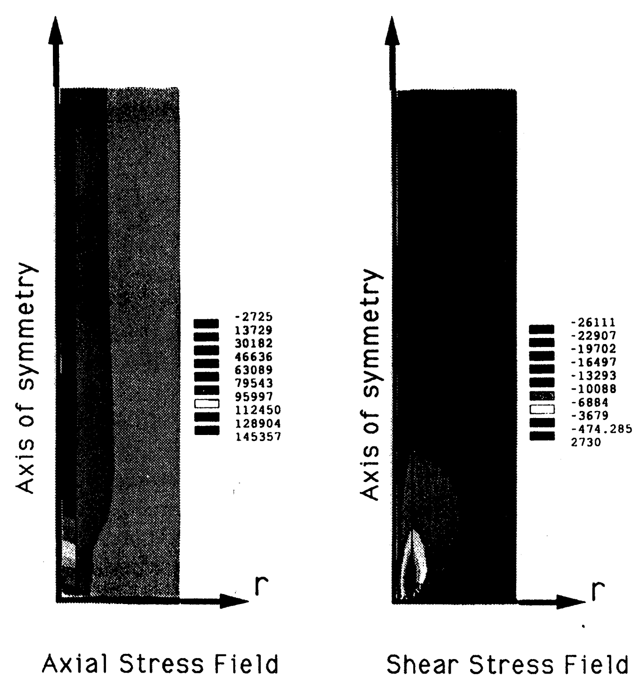

Figure 4. *Stress field (psi) immediately after 80 ksi (551 MPa) axial stress applied.*

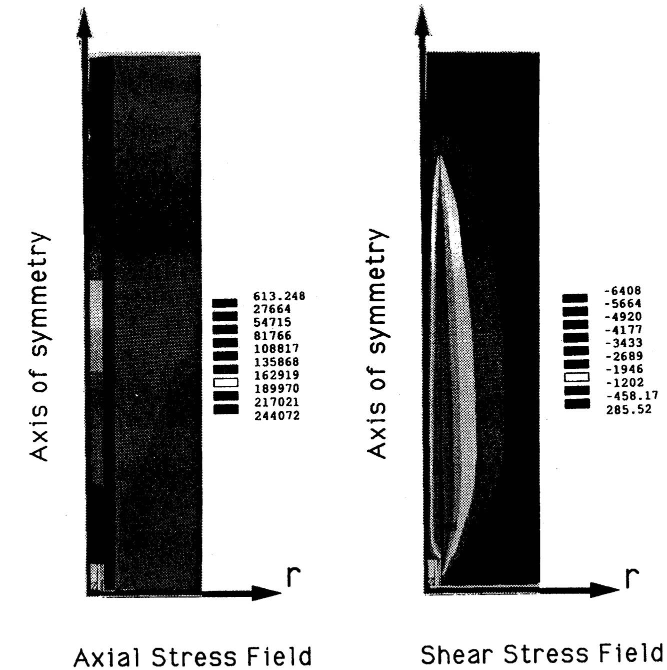

Figure 5. *Stress field (psi) after 80 ksi (551 MPa) stress applied for 4000 hours.*

ratio increases from 2 to 11.8 over the 4000 hour period. The plots of the fiber bundle stress and δ/D over the 4000 hour period are shown in Figure 6.

Implicit in the estimation of δ as a function of time is the assumption that δ is not a function of average fiber length within the composite. As the stress on the fibers increases, damage in the form of fiber failure accumulates. This damage can be modeled by decreasing the radius (and volume) of the RVE appropriately. However, as long as the stress field disturbances caused by fiber breaks do not overlap significantly, there is no reason to believe that the ineffective length should be a function of accumulating damage. Therefore, a numerical simulation using a constant RVE volume can be used to predict the time-dependent value of δ.

DISCUSSION

The fiber bundle strength, σ_c, is easily calculated as a function of δ and the known initial average fiber length, L_i, using Equation (14). The bundle strength for a range of initial fiber lengths are plotted on Figure 7.

The fiber bundle stress–δ combination existing during the 4000 hour period for the case when only coulomb friction provides shear transfer at the fiber-matrix interface is plotted as curve A of Figure 7.

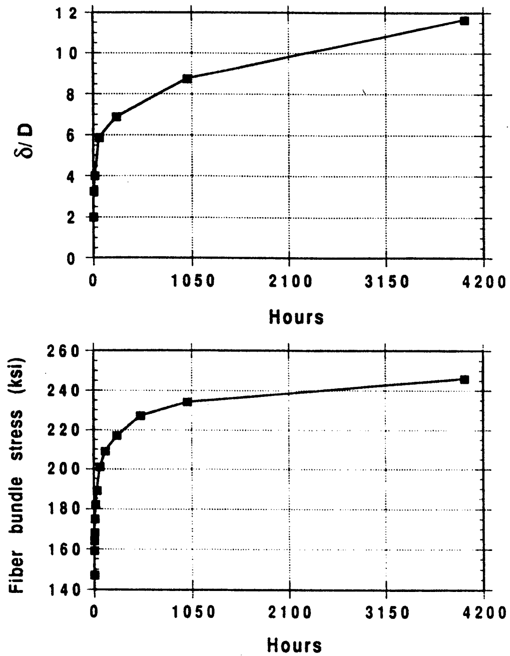

Figure 6. Fiber bundle stress and δ/D versus time (D = 20 μm).

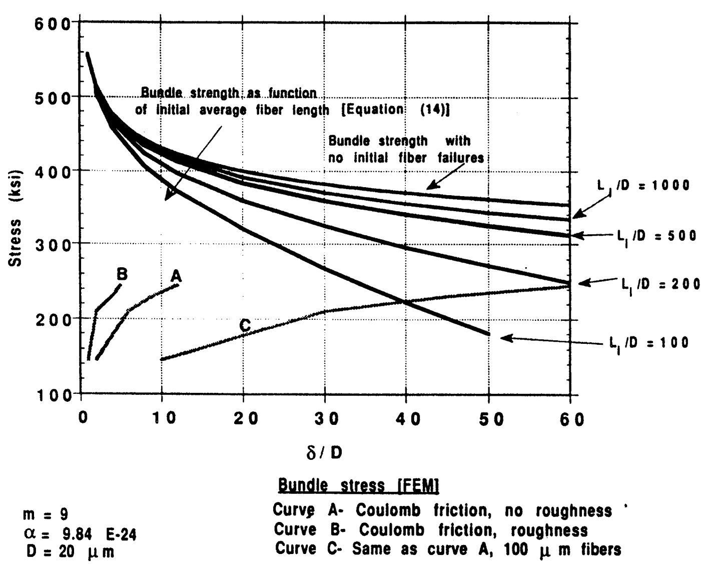

Figure 7. Bundle strength and bundle stress.

The growth in δ evident in Figure 6 results because the stress on the fibers increases and because compressive stresses at the interface decrease. When an additional parameter such as interface roughness is included, growth of δ over time is less dramatic. Interface roughness can be modeled as a constant term which, when added to the term associated with coulomb friction, increases the shear stress threshold required before slip at the interface can occur. When an interface roughness constant of 20 ksi is added to the model, the value of δ/D after 4000 hours grew only to a value of four. Curve B on Figure 7 plots the σ_b-δ combination when interface roughness is included.

The ultimate strength of a composite using fibers with a larger diameter or a weaker bond strength (lower μ) can be extrapolated as follows. A simple force balance on the fiber in the axial direction over the ineffective length, δ, shows that the value of δ should be proportional to the fiber diameter and inversely proportional to μ. If this assumption is correct, then the ineffective length of a composite with 100 μm diameter fibers will be five times greater than a composite with 20 μm diameter fibers. Similarly, if $\mu = 0.2$ instead of 1.0, a fivefold increase in the value of δ will result. The model shown in Figure 2 was used to verify these dependences of δ with both fiber diameter and μ. Curve C on Figure 7 shows the fiber bundle stress–δ plot which would exist over the 4000 hour period of load application if either 100 μm fibers ($m = 1.0$) or 20 μm fibers ($m = 0.2$) were used.

Changing the fiber diameter (for a constant fiber volume fraction), the coefficient of friction, μ, or adding parameters such as surface roughness, change the time-dependent value of δ. However, the value of the bundle stress, σ_b, is unaffected by δ and can always be accurately predicted using analytic models [13].

What information is provided by Figure 7? The ratio of bundle strength [Equation (14)] to bundle stress (either FEM or analytic model) provides the factor-of-safety. Whenever the fiber bundle stress exceeds the bundle strength, the fibers (and composite) will catastrophically fail. Parameters such as the fiber diameter, the interface shear strength (assumed proportional to μ), or the initial average fiber length, L_i, control the gap between the fiber bundle strength and bundle stress. The relative importance of different parameters can be determined. For example, if a high shear stress can be transmitted across the fiber-matrix interface, and the growth of the ineffective length is insignificant over time, then either a large fiber diameter, or a high initial fiber break density may be acceptable. However, a weak interface coupled with a relatively short initial average fiber length may result in failure at a surprisingly low applied composite stress.

CONCLUSIONS

The time-dependent value of the ultimate strength of a CFMMC can be predicted by the model presented. The fiber-matrix interface properties, the initial fiber break density, and the fiber diameter are all parameters which can significantly affect the bundle strength. A graph, as shown in Figure 7, can be used to understand the relative importance of these parameters and to estimate the reduction of bundle strength and factor-of-safety over time.

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APPENDIX 1

Properties of Ti 6-4 at 800°F

Elastic Modulus [15,16]	:	12.7 msi
Poisson Ratio [15]	:	.3
CTE [16]	:	5.8 E-6/F
Shear Modulus [15]	:	5.0 msi
Yield Strength [15,16]	:	72 ksi
Creep Equation [14]	:	$d\epsilon/dt = A(e^{-B/T})(t^{-C})(\sigma^{D})$

A = .083 hrs$^{-.366}$ psi$^{-1.715}$
B = 30,000 F
C = −.634
D = 1.715
T = 1260 R
t = time (hours)
σ = stress (psi)

Properties of Interface

Coefficient of friction	:	1.0

Properties of Alumina Fibers at 800°F

Elastic Modulus [18]	:	55 msi
Poisson Ratio [18]	:	.27
CTE [17]	:	4.6 E-6/F
Shear Modulus	:	21.6 msi
Average Strength [18]	:	400 ksi*
Weibull Modulus [18]	:	9

APPENDIX 2

Bundle Length Selection in Statistically-Based Composite Strength Models

MOTIVATION

Rosen [1] assumes the strength of fibers in a composite is equal to the strength of a dry bundle whose length is the ineffective length, δ. Physical arguments can be used to defend the choice of δ as the approximate length scale which characterizes the strength of fibers in a composite. However, it is not obvious that δ (as opposed to 2δ, for example) is the bundle length choice which best represents the strength of fibers in a composite. If the Weibull modulus is relatively low, the predicted fiber strength changes significantly when the bundle length used to represent the fiber strength is doubled. Obviously, choosing the correct length is important. In addition to the arbitrary selection of δ as the appropriate bundle length. Rosen also neglected the load-carrying capability of broken fibers which exists in a composite. The contribution from these broken fibers can alter the predicted effective strength.

*Strength based on one inch gauge length.

Curtin [19] has also developed a model which uses the length scale δ to predict composite strength. Curtin assumes the "correct" bundle length is 2δ. Intuitively, this selection makes more sense because a fiber failure "communicates" with other fiber failures within distance δ in either direction. Curtin also includes the load-carrying capability of broken fibers in his prediction of fiber strength. Curtin, however, does not rigorously justify his use of the 2δ-long bundle. In addition, his method to include the load-carrying contribution from broken fibers only generates a correct answer when the assumed bundle length is 2δ.

Rosen's and Curtin's predictions are derived after initial assumptions are made concerning the appropriate bundle length. This appendix presents a method to determine the bundle strength and critical damage as a function of bundle length. A criteria is proposed which states that the bundle length which maximizes damage prior to failure is the "correct" choice. Therefore, the ability to determine which method is "most" appropriate can be determined.

ANALYSIS OF 2δ-LONG BUNDLE

Assume, initially, that 2δ is the correct bundle length. The aim of the following analysis is to compute the average fiber stress on axial planes $\pm\delta$ shown in Figure 8 just prior to failure. These axial planes form the boundary of the volume which encompasses a bundle $\pm 2\delta$ in length. Unbroken and broken fibers are shown as well as assumed axial stress profiles within the broken fibers between axial planes $\pm\delta$. The fiber bundle strength differs from the strength predicted by Equation (8) (with $\delta = 2\delta$) because broken fibers carry load.

The average fiber stress at planes $\pm\delta$ can be calculated by considering only the axial span between $\pm 2\delta$. The fibers shown in Figure 8 can be divided into three types. Fibers can either be broken within the $\pm\delta$ region shown, broken between $\pm 2\delta$ but not between $\pm\delta$, or not broken between $\pm 2\delta$. At axial locations $\pm\delta$, the fibers which are not broken between $\pm 2\delta$ carry the nominal stress corresponding to unbroken fibers. The stress in the remaining two types of fibers at either $\pm\delta$ can vary from zero (when a break occurs very near either $\pm\delta$) to full load (when a break occurs farther than δ away from an axial plane). As shown on Figure 8, the axial stress is assumed to increase linearly from a value of zero at the break to the nominal value existing in unbroken fibers over length δ. Any other distribution can be easily incorporated.

Five fiber breaks are shown in Figure 8. Consider the three broken fibers shown inside the $\pm\delta$ wide region. The axial stress carried at top and bottom surfaces of the $\pm\delta$ region by the fiber broken exactly at the middle of the $\pm\delta$ wide

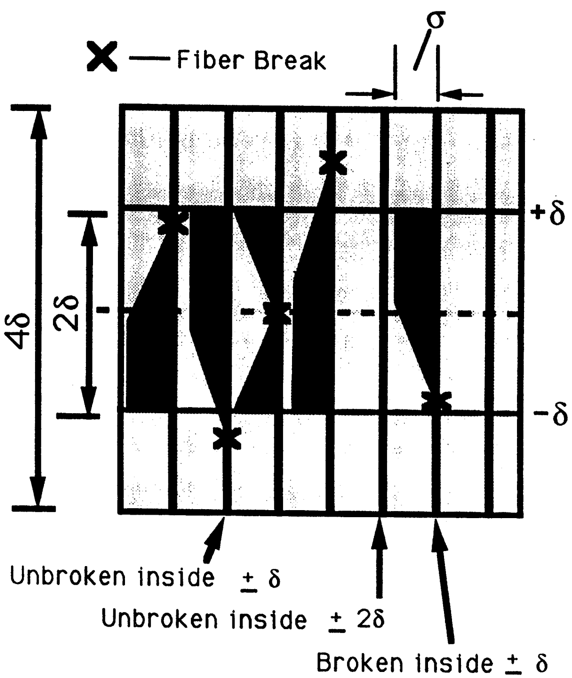

Figure 8. *Model to calculate effect of broken fibers on bundle strength on composites.*

region is equivalent to that being carried by unbroken fibers. The axial stress in the remaining two fibers is different at each end of this region. In both cases shown, the stress being carried at one end of the $\pm\delta$ region is almost zero, while at the other end, the stress is equal to the stress carried by unbroken fibers. These two fibers can be considered a couple and the sum of the load carried at each end by the fiber pair is one-half the load which would be carried by two unbroken fibers. A fiber break pair is defined as two breaks which are on opposite sides and equidistant from the plane shown bisecting Figure 8 and perpendicular to the longitudinal axis. Then, as the distance between the pair decreases from a maximum value of 2δ to a minimum value of zero, the average axial stress of the pair at planes $\pm\delta$ increases from a value one-half the stress existing in unbroken fibers to the full value. The distance between break pairs in the longitudinal direction is random. Therefore, the average stress at $\pm\delta$ within all the fibers broken between $\pm\delta$ is three-quarters of the stress existing in the unbroken fibers.

The same number of broken fibers exist outside of $\pm\delta$ but inside $\pm 2\delta$ as there are fibers which are broken inside the $\pm\delta$ region. The two fibers shown broken outside $\pm\delta$ in Figure 8 can also be considered a couple. It can be argued that fibers which fail outside $\pm\delta$ but inside $\pm 2\delta$ carry, on average, three-quarters the load of unbroken fibers at the axial positions of $\pm\delta$. Therefore, although this

group of fibers is unbroken between $\pm\delta$, their average stress at $\pm\delta$ is one-quarter less than the nominal unbroken value.

The contributions from broken fibers on planes at $\pm\delta$ can now be summed. A number of fibers, broken between $\pm\delta$, carries three-quarters the load carried by unbroken fibers. An equal number of fibers broken outside $\pm\delta$ carry a stress one-quarter less at the $\pm\delta$ boundary than would be carried by unbroken fibers. An equivalent load-carrying capability from the broken fibers results if each fiber broken between $\pm\delta$ is considered to carry one-half the load carried by unbroken fibers while simultaneously all fibers broken outside $\pm\delta$ are considered to carry the full load. By redistributing the load in such a fashion, the total load carried by the fibers can be estimated by Equation (A1) shown below.

$$(\sigma_b)(A_t) = (\sigma)(A_{unb}) + (\sigma_{br})(A_{br}) \tag{A1}$$

where

σ_b = fiber bundle stress (average stress)
σ = stress in fibers considered unbroken
σ_{br} = average stress in fibers broken between $\pm\delta$ at axial positions $+\delta$
A_t = total fiber cross-sectional area
A_{unb} = total unbroken fiber cross-sectional area
A_{br} = total broken fiber cross-sectional area

The cross-sectional area which can be attributed to the broken and unbroken fibers at a given stress for a bundle of length 2δ [Equation (1)] can be determined and is shown below.

$$A_t = N_f A_f \tag{A2}$$

$$A_{unb} = N_f A_f[\exp(-2\delta\alpha\sigma^m)] \tag{A3}$$

$$A_{br} = N_f A_f[1 - \exp(-2\delta\alpha\sigma^m)] \tag{A4}$$

where

A_f = cross-sectional area of a fiber
N_f = the total number of fibers

We have already determined that fibers which are broken between $\pm\delta$ can be considered to carry one-half the load carried by the unbroken fibers. Therefore, by combining Equations (A1)–(A4), the bundle stress is given below as a function of the stress in the unbroken fibers.

$$\sigma_b = (\sigma)\left[\frac{1}{2} + \frac{1}{2}\exp(-2\delta\alpha\sigma^m)\right] \tag{A5}$$

Table 1. C as function of m.

m	C
10	.445
8	.43
6	.40
4	.307

The bundle strength is calculated by maximizing Equation (A5) with respect to the unbroken fiber stress, σ. A maximum exists when Equation (A6) is satisfied.

$$\frac{1 + \exp(-2\delta\alpha\sigma^m)}{\exp(-2\delta\alpha\sigma^m)} = 2\delta\mu\alpha\sigma^m \tag{A6}$$

The maximum stress, σ_m, in unbroken fibers in a composite differs from the maximum stress in unbroken fibers in a dry bundle of length 2δ. It is convenient to characterize σ_m as the maximum unbroken fiber stress in a dry bundle of length 2δ [Equation (6) with $L = 2\delta$] multiplied by a constant of proportionality ($C^{-1/m}$).

$$\sigma_m = (2C\delta\alpha m)^{-1/m} \tag{A7}$$

When combined, Equations (A6) and (A7) produce the following equality:

$$C\left[\exp\left(\frac{-1}{m}\right)\right]^{-1/C} + C = 1 \tag{A8}$$

The values of C which satisfy Equation (A8) as a function of the Weibull modulus, m, are given in Table 1.

We can now determine the bundle strength predicted by Equation (A5) by inserting the appropriate value of σ_m, provided by Equation (A7) and Table 1. Examine first, the case if $m = 10$. If Equations (A7) and (A5) are combined, the bundle strength in the composite is given by Equation (A9).

$$\sigma_c = (.974)(2\delta\alpha m)^{-1/m} \tag{A9}$$

This value can be compared to the bundle strength of a dry bundle of length 2δ. The strength of a dry bundle of length 2δ [Equation (7)] is provided by Equation (A10).

$$\sigma_c = (2\delta\alpha m)^{-1/m} \exp\left[\frac{-1}{m}\right] \tag{A10}$$

The strength of a 2δ-long dry bundle is given by Equation (A11) when $m = 10$.

$$\sigma_c = (.9048)(2\delta\alpha m)^{-1/m} \tag{A11}$$

The strength of the 2δ-long bundle in a composite is 7.72% stronger than the dry bundle of the same length [calculated from the ratio of Equations (A9) and (A11)]. When $m = 10$, a dry bundle of length δ is 7.2% stronger than a dry bundle of length 2δ. Therefore, if $m = 10$, the fiber bundle strength is accurately characterized either by a dry bundle of length δ or by bundle of length 2δ where the effects of the broken fibers must be considered. How general is this result? What if the value of m is much smaller than 10?

The same analysis is repeated for the case where $m = 4$. In this case, the strength of the 2δ-long bundle in a composite is 24.5% stronger than the dry bundle strength of the same length. A dry bundle of length δ is 19% stronger than a dry bundle of length 2δ. Therefore, even when $m = 4$, composite strength is still well-approximated by the strength of a dry bundle of length δ.

The same analysis used to generate Equation (A5) could be used to determine the fiber bundle strength based on the assumption that a bundle length other than 2δ is appropriate. For example, if a bundle length of δ is chosen, Figure 8 would then be a total of 3δ in length. If the load recovery profile at a break location is linear, it can be shown that the effect of fiber breaks outside the central δ-wide region on the average stress at the bundle boundary is equal and opposite to the effect of breaks within the δ-wide region. Therefore, when the bundle length is equal to δ, the net effect of the load-carrying capability of the fibers is zero and the dry bundle prediction is identical to the prediction of the fiber strength in the composite.

BUNDLE LENGTH SELECTION

The prediction of fiber strength in a composite does not vary much if the composite is modeled either as a series of 2δ-long bundles in which the contributions from broken fibers are included, or as dry δ-long bundles. However, the academic question remains, "What is the best choice?"

The authors propose that the representative volume which results in the maximum damage in the composite prior to failure is the best choice. Each fiber failure releases energy. The composite tends towards its lowest energy state, which coincides with maximum damage. Therefore, the bundle length which maximizes the density of fiber breaks in the composite is the bundle length which best characterizes the response of the composite to an applied load.

Assume a composite is subjected to a strain sufficient to break fibers. If the fibers are viewed as a series of fiber bundles, then a percentage of fibers, $F(\sigma)$, within each bundle will break at stress σ [Equation (1)]. The total number of breaks within the composite, N_b, is given by Equation (A12) below:

$$N_b = N\left[\frac{L}{L_b}\right] F(\sigma) \tag{A12}$$

N = number of fibers in cross section of specimen
L = length of specimen
L_b = length of bundle

The value of $F(\sigma)$ is a function of the bundle length chosen. Its value can be determined as follows: for any assumed bundle length, an appropriate version of Equation (A5) can be derived. The unbroken fiber stress, σ, and $F(\sigma)$ can be calculated if the applied bundle stress, σ_b, is given. The value of $F(\sigma)$ increases monotonically with bundle length but the number of fiber breaks inside the composite does not.

Assuming a linear recovery profile, we use this criteria to compare whether a bundle length of δ or 2δ is more appropriate. When the bundle length of 2δ is chosen and assuming that $m = 10$, one calculates that 22.1% of the fibers break in the 2δ-long bundle prior to failure [combine Equations (1) and (A9) and Table 1]. As mentioned earlier, if the analysis which generated Equation (A5) was applied to a bundle length of δ, then the predicted strength of the fibers in the composite is equal to the strength of a dry bundle of length δ and 9.5% of the fibers within a dry bundle of any length break prior to failure (if $m = 10$). Therefore, over a 2δ region, 19% of the fibers can be considered broken. The best bundle length choice is 2δ because it results in more damage prior to failure. In fact, if the analysis methodology used to generate Equation (A5) is applied to any bundle length other than 2δ, the accumulated damage prior to failure will decrease. Therefore, assuming a linear load recovery profile at a fiber break, the 2δ-long bundle is the "correct" choice. If a stress recovery profile differs substantially from the linear profile assumed, then a different optimal bundle length may result. The choice of the 2δ-long bundle also happens to maximize strength. It is interesting that the choice which maximizes strength also maximizes damage since damage typically reduces strength.

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