On the Mechanics of Thin-Walled Laminated Composite Beams

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ABSTRACT: A formal engineering approach of the mechanics of thin-walled laminated beams based on kinematic assumptions consistent with Timoshenko beam theory is presented. Thin-walled composite beams with open or closed cross section subjected to bending and axial load are considered. A variational formulation is employed to obtain a comprehensive description of the structural response. Beam stiffness coefficients, which account for the cross section geometry and for the material anisotropy, are obtained. An explicit expression for the static shear correction factor of thin-walled composite beams is derived from energy equivalence. A numerical example involving a laminated I-beam is used to demonstrate the capability of the model for predicting displacements and ply stresses.

1. INTRODUCTION

ADVANCED MATERIALS, MAINLY fiber reinforced plastic (FRP) composites, will partially replace conventional materials in civil engineering type structures (Barbero and GangaRao [1]). Most recent applications in transportation systems, offshore structures, chemical facilities and communication systems, show the usefulness of composite structures like thin-walled beams and columns. Compared to standard construction materials, composite materials present many advantages, e.g., light weight, corrosion resistance, and electromagnetic transparency. Most prominent is the property of tailoring the material for each particular application. Structural properties like stiffness, strength, and buckling resistance depend on the material system (composite) and the shape of the cross-section of the member. Like with steel structural shapes, it is possible to optimize the section to increase the bending stiffness without compromising the maximum bending strength. Unlike steel shapes, with composite beams it is possible to optimize the material itself by choosing among a variety of resins, fiber systems, and fiber orientations. Changes in the geometry can be easily related to changes in the bending stiffness through the moment of inertia. Changes in the material do not lead to such obvious results, because composites have properties that not only depend on the orientation of the fibers but also exhibit modular ratios that could differ considerably from usual values in conventional isotropic materials (Barbero [2]).

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Although beams and columns are the most commonly used structural elements, the theory of laminated beams has been less developed than the theory of laminated plates. Laminated beam theories were initially derived as extensions of existing plate or shell theories. Bert and Francis [3] presented a comprehensive review of the initial beam theories. Berkowitz [4] pioneered a theory of simple beams and columns for anisotropic materials. Vinson and Sierakowski [5] applied classical lamination theory along with a plane strain assumption to obtain the extensional, coupling and bending stiffness for an Euler-Bernoulli type laminated beam (A_{11},B_{11},D_{11}). A theory for orthotropic thin-walled composite beams was proposed by Bank and Bednarczyk [6], where the in-plane material properties were obtained using classical lamination theory or coupon tests. A Vlasov theory for thin-walled open cross sections composed of plane symmetric laminates was proposed by Bauld and Tzeng [7] disregarding shear strains in the middle planes. Massonnet [8] addressed the problem of warping in a transversely isotropic beam by complementing a mechanics of materials approach with corrective terms derived using theory of elasticity. Bauchau [9] and Bauchau et al. [10] provided a more comprehensive treatment to the problem of warping by using variational principles to model anisotropic thin-walled beams with closed cross sections. A general finite element with 10 degrees of freedom per node was derived by Wu and Sun [11] for thin-walled laminated composite beams by modifying the assumptions of the Vlasov theory. Skudra et al. [12] proposed a theory for thin-walled symmetrically laminated beams of open profile, and they illustrated the distribution of forces in a flat homogeneous anisotropic strip. Tsai [13] defined engineering constants from the laminate compliances, and employed them to obtain deflections for laminated beams. He further employed laminated plate theory to determine ply stresses. In the present work, kinematic assumptions consistent with the Timoshenko beam theory are employed in order to generate beam stiffness coefficients. A distintictive feature of the present approach with respect to existing formulations [7,9,10,12] is the possibility of considering not only membrane stresses but also flexural stresses in the walls. This assumption seems to be more appropriate for moderately thick laminated beams employed in civil engineering–type structures.

The bending extension coupling that may result from material and/or geometric asymmetry is usually taken into account by bending-extension coupling stiffness coefficients. In this work, the position of the neutral axis is defined in such a way that the behavior of a thin-walled beam-column with asymmetric material and/or cross-sectional shape is completely described by axial, bending, and shear stiffness coefficients (A_z,D_y,F_y) only.

While the importance of considering a consistent shear coefficient in the

Timoshenko beam theory for anisotropic beams was recognized early [4], a comprehensive treatment in the framework of a formal mechanics approach is not available. Cowper [14] derived a shear coefficient for isotropic materials from the elasticity solution of the classical Saint Venant flexure problem under the assumption of linearly varying shear force. While flexure functions are available for regular sections (Love [15]), in the case of thin-walled sections this approach requires the evaluation of the shear stress distribution from mechanics of materials methods. Dharmarajan and McCutchen [16] extended the formulation of Cowper for orthotropic beams without addressing the case of thin-walled sections. Bank [17], applying the same equations as those proposed by Dharmarajan and McCutchen [16], presented a derivation based on the work of Cowper [14] for the case of thin-walled beams restricted to assemblies of horizontal and vertical orthotropic panels. Bank and Melehan [18] further extended the formulation to multicelled thin-walled sections. Bert [19] presented a derivation of the static shear factor for beams of nonhomogeneous cross section. He considered rectangular beams with layers perpendicular to the plane of bending. Tsai et al. [20] derived a shear correction factor for rectangular laminates subjected to torsion. In the present work, the shear correction factor is obtained from energy equivalence as in References [19] and [20]. The derivation of the shear factor is based on computing the shearing stress distribution in the cross section.

The objective of this article is to present the derivations of the Mechanics of thin-walled Laminated Beams (MLB) for open and closed cross sections. A variational formulation is employed to obtain a comprehensive description of the structural response of composite beams subjected to bending and axial load. The example presented, involving a laminated I-beam, illustrates the capability of the model for predicting displacements and ply stresses, while envisioning the potential of the approach for the design optimization of new structural shapes.

2. STRUCTURAL MODEL OF THE BEAM SUBJECTED TO FLEXURE

2.1 Geometry and Loading Definition

A straight thin-walled composite beam-column with one axis of geometric and material symmetry will be considered. We define a Cartesian coordinate system (x,y,z), with the z-axis parallel to the axis of the beam and one of the other transverse axes orthogonal to the plane of symmetry. The beam, made of assembled flat walls, could have either an open or closed cross section. The middle surface of the beam cross section is represented by a polygonal line called the contour. We introduce for each wall a local contour coordinate system (s_i,n_i,z) placed on the middle surface of the wall, where the axes s_i and n_i are tangent and normal to the contour respectively (see Figure 1). The contour is defined parametrically by the stepwise linear functions $x(s_i)$ and $y(s_i)$. The orientation of the ith wall is characterized by the angle ϕ_i as follows

$$\frac{dx}{ds_i} = \cos \phi_i$$

$$\frac{dy}{ds_i} = \sin \phi_i \tag{1}$$

The transverse loads are applied through the shear center and are contained in a plane normal to one of the principal axes (x,y). Under this loading condition the beam is subjected to symmetrical bending decoupled from torsion. The present derivation is restricted to symmetric bending for sake of brevity but could be easily extended to non-symmetric bending. The joints of the cross section are modelled at the intersection of the walls' middle surfaces. This assumption, as stated by Ng, Cheung, and Bingzhang [21], is amply justified due to the small strain energy contribution of the joints in thin-walled beams.

2.2 Kinematic Assumptions

Following Timoshenko beam theory, the basic assumptions regarding the present mechanics of thin-walled laminated beams are introduced.

Assumption 1. The contour does not deform in its own plane. The motions $\bar{u}$ and $\bar{v}$ along the s_i and n_i directions respectively, at a point on the middle surface of the ith wall, can be expressed in terms of the rigid body motions $u(z)$ and $v(z)$ in the x and y directions respectively (see Figure 2).

$$\begin{aligned} \bar{u}(s_i, z) &= u(z) \cos \phi_i + v(z) \sin \phi_i \\ \bar{v}(s_i, z) &= - u(z) \sin \phi_i + v(z) \cos \phi_i \end{aligned} \tag{2}$$

Considering symmetrical bending normal to the x axis results $u(z) = 0$.

Assumption 2. A plane section originally normal to the beam axis remains

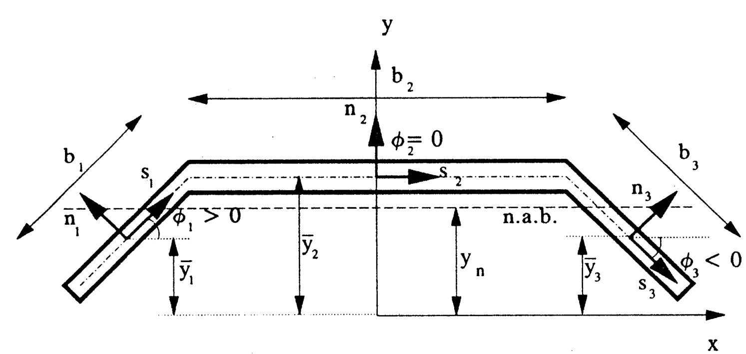

Figure 1. *Cross-section geometry and reference systems.*

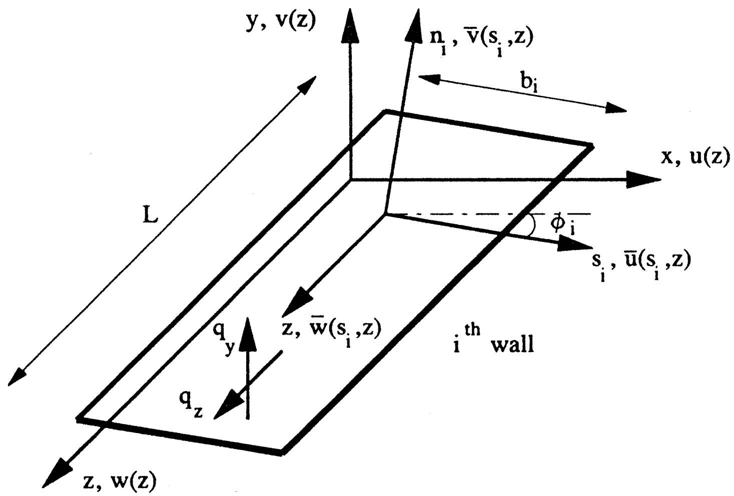

Figure 2. Motions and applied loads.

plane, but not necessarily normal to the beam axis due to shear deformation. The axial displacement of the contour can be expressed as

$$\bar{w}(s_i,z) = w(z) - [y(s_i) - y_n]\psi_y(z) \tag{3}$$

where $w(z)$ represents the axial displacement of the beam in the z direction at the position of the neutral axis of bending y_n. The kinematic variable $\psi_y(z)$ measures the rotation in the plane of bending. This assumption could be modified to account for residual displacements or warping of the cross section by considering additional corrective terms. The out-of-plane warping could be expressed as a series expansion in terms of a set of orthogonal functions which depend upon the cross-sectional geometry and the composite lay-up, and a set of new kinematic variables that account for the loading and the boundary conditions. In this sense, Hjelmstad [22] obtained the warping functions for isotropic materials from the exact solution of the Saint Venant's flexure problem through the application of the Gram-Schmidt orthogonalization process. Bauchau [9] derived from energy principles the eigenwarping functions for the case of curvilinear orthotropic materials.

3. ANALYSIS OF A FLAT LAMINATED WALL AS A BEAM COMPONENT

3.1 Constitutive Equations

Employing Classical Lamination Theory (CLT), the general constitutive rela-

tions for a laminated wall with respect to the local contour coordinate system depicted in Figure 2 are

$$\begin{Bmatrix} \{\bar{N}\} \\ \{\bar{M}\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] \\ [B] & [D] \end{bmatrix} \begin{Bmatrix} \{\bar{\epsilon}\} \\ \{\bar{\kappa}\} \end{Bmatrix} \tag{4}$$

where the laminate resultant forces and moments are

$$\{\bar{N}\} = \begin{Bmatrix} \bar{N}_z \\ \bar{N}_s \\ \bar{N}_{sz} \end{Bmatrix} \qquad \{\bar{M}\} = \begin{Bmatrix} \bar{M}_z \\ \bar{M}_s \\ \bar{M}_{sz} \end{Bmatrix} \tag{5}$$

and the laminate strains and curvatures are

$$\{\bar{\epsilon}\} = \begin{Bmatrix} \bar{\epsilon}_z \\ \bar{\epsilon}_s \\ \bar{\gamma}_{sz} \end{Bmatrix} \qquad \{\bar{\kappa}\} = \begin{Bmatrix} \bar{\kappa}_z \\ \bar{\kappa}_s \\ \bar{\kappa}_{sz} \end{Bmatrix} \tag{6}$$

The expressions for the stiffness submatrices $[A]$, $[B]$ and $[D]$ are defined in Jones [23]. By full inversion of the stiffness matrix, Equation (4) results in

$$\begin{Bmatrix} \{\bar{\epsilon}\} \\ \{\bar{\kappa}\} \end{Bmatrix} = \begin{bmatrix} [\alpha] & [\beta] \\ [\beta]^T & [\delta] \end{bmatrix} \begin{Bmatrix} \{\bar{N}\} \\ \{\bar{M}\} \end{Bmatrix} \tag{7}$$

where the compliance submatrices are

$$\begin{aligned} [\alpha] &= [[A] - [B][d][B]]^{-1} \\ [\delta] &= [[D] - [B][a][B]]^{-1} \\ [\beta] &= [\beta]^T = -[a][B][\delta] = -[d][B][\alpha] \end{aligned} \tag{8}$$

and

$$[a] = [A]^{-1} \qquad [d] = [D]^{-1}$$

Tsai [13] employed the elements of the compliance matrices presented in these equations to define in-plane and flexural engineering constants. In this work the matrix Equation (7) for general laminates is reduced for the case of laminates that are components of thin-walled beams. Consistent with beam theory and based on Assumption 1 we consider that for a laminated wall the resultant force and moment originated by the transverse normal stresses (in the s_i direction) are negligible, then

$$\bar{N}_s = \bar{M}_s = 0 \tag{9}$$

Wu and Sun [11] showed that for slender thin-walled laminated beams without ribs Equation (9) yields more accurate natural frequencies than the alternative plane strain assumption ($\bar{\epsilon}_s = \bar{\kappa}_{sz} = 0$). For bending without torsion we can further state that

$$\bar{M}_{sz} = 0 \tag{10}$$

Incorporating the conditions (9) and (10) into Equation (7) yields for the ith wall

$$\begin{Bmatrix} \bar{\epsilon}_z \\ \bar{\kappa}_z \\ \bar{\gamma}_{sz} \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \beta_{11} & \alpha_{16} \\ \beta_{11} & \delta_{11} & \beta_{16} \\ \alpha_{16} & \beta_{16} & \alpha_{66} \end{bmatrix}_i \begin{Bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{N}_{sz} \end{Bmatrix} \tag{11}$$

It is convenient to derive the governing equations for a thin-walled laminated beam from energy principles. The strain energy per unit length considering the beam as an assembly of n walls results in

$$\bar{U}(z) = \frac{1}{2}\sum_{i=1}^{n}\int_{-b_i/2}^{b_i/2} [(\alpha_{11})_i\bar{N}_z^2 + 2(\beta_{11})_i\bar{N}_z\bar{M}_z + 2(\alpha_{16})_i\bar{N}_z\bar{N}_{sz}$$
$$+ (\delta_{11})_i\bar{M}_z^2 + 2(\beta_{16})_i\bar{M}_z\bar{N}_{sz} + (\alpha_{66})_i\bar{N}_{sz}^2]ds_i \tag{12}$$

For each wall the position of the middle surface is defined by the function

$$y(s_i) = s_i \sin\phi_i + \bar{y}_i \qquad \text{for} \qquad -\frac{b_i}{2} \le s_i \le \frac{b_i}{2} \tag{13}$$

where b_i is the wall width and $\bar{y}_i$ is the position of the wall centroid (see Figure 1). We observe from the expression (12) that the coefficients α_{16} and β_{16} are responsible for the shear-extension and shear-bending coupling respectively. In order to decouple the variational problem, and within the scope of the engineering applications, we restrict our formulation to laminates that satisfy

$$\alpha_{16} = \beta_{16} = 0 \tag{14}$$

These conditions are satisfied by laminates with off-axis plies that are balanced symmetric. Hence Equation (11) reduces to

$$\begin{Bmatrix} \bar{\epsilon}_z \\ \bar{\kappa}_z \\ \bar{\gamma}_{sz} \end{Bmatrix} = \begin{bmatrix} \alpha_{11} & \beta_{11} & 0 \\ \beta_{11} & \delta_{11} & 0 \\ 0 & 0 & \alpha_{66} \end{bmatrix}_i \begin{Bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{N}_{sz} \end{Bmatrix} \tag{15}$$

where the compliance coefficient β_{11} accounts for bending-extension coupling due to unsymmetric orthotropic layers. By inverting the compliance matrix in

Equation (15) we obtain the stiffness matrix of the ith wall of a thin-walled laminated beam as follows

$$\begin{Bmatrix} \bar{N}_z \\ \bar{M}_z \\ \bar{N}_{sz} \end{Bmatrix} = \begin{bmatrix} \bar{A}_i & \bar{B}_i & 0 \\ \bar{B}_i & \bar{D}_i & 0 \\ 0 & 0 & \bar{F}_i \end{bmatrix} \begin{Bmatrix} \bar{\epsilon}_z \\ \bar{\kappa}_z \\ \bar{\gamma}_{sz} \end{Bmatrix} \tag{16}$$

where

$\bar{A}_i = \left(\dfrac{\delta_{11}}{\alpha_{11}\delta_{11} - \beta_{11}^2}\right)_i$ is the extensional stiffness

$\bar{B}_i = \left(\dfrac{-\beta_{11}}{\alpha_{11}\delta_{11} - \beta_{11}^2}\right)_i$ is the bending-extension coupling stiffness

$\bar{D}_i = \left(\dfrac{\alpha_{11}}{\alpha_{11}\delta_{11} - \beta_{11}^2}\right)_i$ is the bending stiffness

$\bar{F}_i = \left(\dfrac{1}{\alpha_{66}}\right)_i$ is the shear stiffness

Therefore the strain energy per unit length [Equation (12)] expressed in terms of the wall stiffness coefficients [Equation (16)] simplifies as follows

$$\bar{U}(z) = \frac{1}{2}\sum_{i=1}^{n}\int_{-b_i/2}^{b_i/2} (\bar{A}_i\bar{\epsilon}_z^2 + 2\bar{B}_i\bar{\epsilon}_z\bar{\kappa}_z + \bar{D}_i\bar{\kappa}_z^2 + \bar{F}_i\bar{\gamma}_{sz}^2)ds_i \tag{17}$$

3.2 Strain-Displacement Relations

The wall strains are derived from the kinematic relations (2) and (3)

$$\begin{aligned} \bar{\epsilon}_z(s_i,z) &= \frac{\partial\bar{w}}{\partial z} = \frac{dw}{dz} - (y(s_i) - y_n)\frac{d\psi_y}{dz} \\ \bar{\gamma}_{sz}(s_i,z) &= \frac{\partial\bar{u}}{\partial z} + \frac{\partial\bar{w}}{\partial s_i} = \left(\frac{dv}{dz} - \psi_y\right)\sin\phi_i \end{aligned} \tag{18}$$

The motions of a point away from the middle surface of the wall follows from Assumption 2. Consequently the wall curvature results in

$$\bar{\kappa}_z(s_i,z) = -\frac{d\psi_y}{dz}\frac{dx}{ds_i} = -\frac{d\psi_y}{dz}\cos\phi_i \tag{19}$$

Equation (19) implies that flexural strains, which vary linearly in the direction of the wall thickness, will be generated in addition to the typically predominant membrane strains [Equation (18)]. Thus, this kinematic model can be applied either to thin or thick laminated walls. Using the strain-displacement relationships (18) and (19), and the parametric definition of the contour [Equation (13)], the strain energy per unit length [Equation (17)] becomes

$$\overline{U}(z) = \frac{1}{2}\sum_{i=1}^{n} b_i\left\{\overline{A}_i\left(\frac{dw}{dz}\right)^2 - 2[\overline{A}_i(\bar{y}_i - y_n) + \overline{B}_i\cos\phi_i]\frac{dw}{dz}\frac{d\psi_y}{dz}\right.$$
$$+\left[\overline{A}_i\left((\bar{y}_i - y_n)^2 + \frac{b_i^2}{12}\sin^2\phi_i\right) + 2\overline{B}_i(\bar{y}_i - y_n)\cos\phi_i + \overline{D}_i\cos^2\phi_i\right]$$
$$\left.\times\left(\frac{d\psi_y}{dz}\right)^2 + [\overline{F}_i\sin^2\phi_i]\left(\frac{dv}{dz} - \psi_y\right)^2\right\} \tag{20}$$

3.3 Ply Strains and Stresses

The axial and shear strains evaluated through the thickness of the ith wall result in

$$\begin{aligned}\epsilon_z(s_i,\xi,z) &= \bar{\epsilon}_z(s_i,z) + \xi\bar{\kappa}_z(s_i,z)\\ \gamma_{sz}(s_i,z) &= \bar{\gamma}_{sz}(s_i,z)\end{aligned} \tag{21}$$

where ξ is the thickness coordinate (in the n_i direction). Although the laminae in a laminated wall are constrained and interact with one another, in order to obtain an approximation for the ply stresses and following Equation (9), we further assume that the transverse normal stresses are insignificant $\sigma_s \approx 0$. This condition yields the following expressions for the axial and shear stresses in the kth layer.

$$\begin{Bmatrix}\sigma_z\\ \sigma_{sz}\end{Bmatrix} = \begin{bmatrix}\hat{Q}_{11} & \hat{Q}_{16}\\ \hat{Q}_{16} & \hat{Q}_{66}\end{bmatrix}^k\begin{Bmatrix}\epsilon_z\\ \gamma_{sz}\end{Bmatrix} \tag{22}$$

where

$$\hat{Q}_{ij} = \overline{Q}_{ij} - \frac{\overline{Q}_{i2}\overline{Q}_{2j}}{\overline{Q}_{22}} \qquad \text{for} \quad i,j = 1,6$$

and $\overline{Q}_{ij}$ are the transformed reduced stiffness coefficients employed in CLT (Jones [23]). For the particular case of a layer with fibers oriented in the direction of the beam axis, the modified stiffness coefficients of Equation (22) reduce to the corresponding lamina elastic constants: $\hat{Q}_{11} = E_1$, $\hat{Q}_{66} = G_{12}$, and $\hat{Q}_{16} = 0$.

4. DERIVATION OF THE BEAM GOVERNING EQUATIONS

4.1 Beam Stiffness Coefficients

The total potential energy to be minimized follows from Equation (20), with the introduction of beam stiffness coefficients, yielding

$$\Pi = \frac{1}{2}\int_0^L\left[A_z\left(\frac{dw}{dz}\right)^2 - 2B_y\frac{dw}{dz}\frac{d\psi_y}{dz} + D_y\left(\frac{d\psi_y}{dz}\right)^2\right.$$
$$\left. + K_yF_y\left(\frac{dv}{dz} - \psi_y\right)^2\right]dz - \int_0^L(q_yv + q_zw)dz \tag{23}$$

where L is the length of the beam, and q_y and q_z are the applied transverse and axial loads respectively (depicted in Figure 2). The beam stiffness coefficients are defined by

$$\begin{aligned}A_z &= \sum_{i=1}^{n}\overline{A}_ib_i\\ B_y &= \sum_{i=1}^{n}[\overline{A}_i(\bar{y}_i - y_n) + \overline{B}_i\cos\phi_i]b_i\\ D_y &= \sum_{i=1}^{n}\left[\overline{A}_i\left((\bar{y}_i - y_n)^2 + \frac{b_i^2}{12}\sin^2\phi_i\right)\right.\\ &\qquad \left. + 2\overline{B}_i(\bar{y}_i - y_n)\cos\phi_i + \overline{D}_i\cos^2\phi_i\right]b_i\\ F_y &= \sum_{i=1}^{n}\overline{F}_ib_i\sin^2\phi_i\end{aligned} \tag{24}$$

The shear correction factor K_y is introduced in order to account for the actual shear stress distribution in the cross section. An expression for K_y based on energy equivalence is derived in this article. The set of equations obtained for the beam stiffness coefficients [Equation (24)], reduces for the case $\phi_i = 0$ to the parallel axis theorem presented by Tsai and Hahn [24]. A reduction to a pure membrane case, where the flexural strains in the wall are negligible, is obtained by setting $\overline{B}_i = \overline{D}_i = 0$.

4.2 Equilibrium Equations

We define the position of the neutral axis of bending of the cross section by setting $B_y = 0$, which yields

$$y_n = \frac{\sum_{i=1}^{n} (\bar{y}_i \bar{A}_i + \cos \phi_i \bar{B}_i) b_i}{A_z}, \tag{25}$$

Introducing the coordinate $y' = y - y_n$, we are able to decouple the extensional and bending responses in Equation (23), as follows.

$$\Pi = \frac{1}{2} \int_0^L \left[A_z \left(\frac{dw}{dz}\right)^2 + D_y \left(\frac{d\psi_y}{dz}\right)^2 + K_y F_y \left(\frac{dv}{dz} - \psi_y\right)^2 \right] dz - \int_0^L (q_y v + q_z w) dz \tag{26}$$

This is done to simplify the formulation along the lines of classical structural analysis as used by the majority of structural engineers. The Timoshenko beam solution is obtained by minimizing the total potential energy [Equation (26)] with respect to the functions w, v, ψ_y. Integrating by parts and applying the fundamental lemma of calculus of variations we obtain the equilibrium equations

$$\begin{gathered} \frac{d}{dz}\left(A_z \frac{dw}{dz}\right) + q_z = 0 \\ \frac{d}{dz}\left[K_y F_y \left(\frac{dv}{dz} - \psi_y\right)\right] + q_y = 0 \\ \frac{d}{dz}\left(D_y \frac{d\psi_y}{dz}\right) + K_y F_y \left(\frac{dv}{dz} - \psi_y\right) = 0 \end{gathered} \tag{27}$$

and the boundary conditions of the system

$$\begin{gathered} A_z \frac{dw}{dz} \delta w |_0^L = 0 \\ D_y \frac{d\psi_y}{dz} \delta \psi_y |_0^L = 0 \\ K_y F_y \left(\frac{dv}{dz} - \psi_y\right) \delta v |_0^L = 0 \end{gathered} \tag{28}$$

4.3 Constitutive Equations

Expressing the total potential energy Equation (26) in terms of the wall stress resultants leads to the following definitions for the beam resultant forces and moments

$$\begin{aligned} N_z(z) &= \sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_z ds_i \\ M_y(z) &= \sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} [\bar{N}_z y'(s_i) + \bar{M}_z \cos \phi_i] ds_i \\ V_y(z) &= K_y \sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_{sz} \sin \phi_i ds_i \end{aligned} \tag{29}$$

and for the generalized beam strains

$$\begin{aligned} \epsilon_z^o(z) &= \frac{dw}{dz} \\ \varkappa_y(z) &= -\frac{d\psi_y}{dz} \\ \gamma_{yz}(z) &= \frac{dv}{dz} - \psi_y \end{aligned} \tag{30}$$

Therefore the beam constitutive equations can be expressed as

$$\begin{aligned} N_z(z) &= A_z \epsilon_z^o \\ M_y(z) &= D_y \varkappa_y \\ V_y(z) &= K_y F_y \gamma_{yz} \end{aligned} \tag{31}$$

For the ith wall, the strains [Equation (18)] and the curvature [Equation (19)] in terms of the beam resultant forces and moments become

$$\begin{aligned} \bar{\epsilon}_z(s_i,z) &= \frac{N_z}{A_z} + y'(s_i) \frac{M_y}{D_y} \\ \bar{\varkappa}_z(s_i,z) &= \frac{M_y}{D_y} \cos \phi_i \\ \bar{\gamma}_{sz}(s_i,z) &= \frac{V_y}{K_y F_y} \sin \phi_i \end{aligned} \tag{32}$$

Given the bending and axial stress resultants for a certain axial position z of the beam, the neutral axis of combined bending and axial force, i.e., the axis for which $\bar{\epsilon}_z$ is zero, follows from Equation (32)

$$y_n^*(z) = y_n - \frac{D_y}{A_z}\frac{N_z(z)}{M_y(z)} \tag{33}$$

5. EVALUATION OF SHEAR STRAIN EFFECTS

5.1 Shearing Stress Distribution

The shearing stress distribution (shear flow) in the cross section of the thin-walled beam is obtained herein from equilibrium in each wall, in terms of the axial stress resultant $\bar{N}_z$. Thus the shear flow evaluated in each wall ($\bar{N}_{sz}^*$) constitutes a refinement over the laminate shear stress resultant ($\bar{N}_{sz}$) calculated from constitutive Equations (16). From Equations (16) and (32) follows

$$\bar{N}_z(s_i,z) = \frac{N_z}{A_z}\bar{A}_i + \frac{M_y}{D_y}[\bar{A}_i y'(s_i) + \bar{B}_i \cos\phi_i] \tag{34}$$

The in-plane equilibrium equation for the ith laminated wall, in the absence of body forces, in the z direction is

$$\bar{N}_{z,z} + \bar{N}^*_{sz,s_i} = 0 \tag{35}$$

Furthermore, in beam theory the following resultant equilibrium equations are employed

$$\frac{dM_y}{dz} = V_y$$
$$\frac{dN_z}{dz} = 0 \tag{36}$$

Substituting Equation (34) in the wall equilibrium Equation (35), and accounting for the beam equilibrium conditions (36), we obtain the shear flow variation in the ith wall

$$\bar{N}^*_{sz,s_i}(s_i,z) = -\frac{V_y}{D_y}[\bar{A}_i y'(s_i) + \bar{B}_i \cos\phi_i] \tag{37}$$

For a cross section having a free edge, or having some other point such that $\bar{N}_{sz}^*(-b_1/2,z) = 0$, we can integrate over the contour up to the rth wall, yielding

$$\bar{N}_{sz}^*(s_r,z) = -\frac{V_y}{D_y}\left[S_r^y + \frac{1}{2}\bar{A}_r \sin\phi_r\left(s_r^2 - \frac{b_r^2}{4}\right) + (\bar{A}_r\bar{y}_r' + \bar{B}_r\cos\phi_r)\left(s_r + \frac{b_r}{2}\right)\right] \tag{38}$$

$$\text{for} \quad -\frac{b_r}{2} \le s_r \le \frac{b_r}{2}$$

where

$$S_r^y = \sum_{i=1}^{r-1}[\bar{A}_i\bar{y}_i' + \bar{B}_i\cos\phi_i]b_i$$

is the weighted static moment of the portion of the cross section corresponding to the first $r - 1$ walls, and $\bar{y}_i' = \bar{y}_i - y_n$. However, for a general closed cross section the shearing stress distribution cannot be obtained employing Equation (38) alone, since we do not know *a priori* where $\bar{N}_{sz}^*$ vanishes. The procedure for a one-cell cross section is to generate free edges by introducing a slit in the compartment, and then close it again by obtaining the shear flow in the compartment that produces zero unit angle of twist. Applying Bredt's formula for thin-walled hollow beams (Cook and Young [25]), we can write for a compartment composed of n' walls

$$\sum_{i=1}^{n'}\frac{1}{\bar{F}_i}\int_{-b_i/2}^{b_i/2}\bar{N}_{sz}^*(s_i,z)ds_i = 0 \tag{39}$$

The net shear flow is $\bar{N}_{sz}^*(s_i,z) = [\bar{N}_{sz}^*(s_i,z)]_{\text{open}} + \bar{N}_{sz}^o$ where $[\bar{N}_{sz}^*]_{\text{open}}$ is the variable open-cell shear flow obtained from Equation (38), and $\bar{N}_{sz}^o$ is the uniform closed-cell shear flow released by the slit. For multicell cross sections the above procedure has to be repeated by satisfying Equation (39) for each compartment. Therefore, the corresponding shearing strain distribution in the ith wall is

$$\bar{\gamma}_{sz}^*(s_i,z) = \frac{1}{\bar{F}_i}\bar{N}_{sz}^* \tag{40}$$

5.2 Location of the Shear Center

The location of the shear center $S(x_s,y_s)$ is defined in order to decouple bending and torsion. For a cross section having one axis of symmetry, one of the coordinates of the shear center is known. Hence the other coordinate is obtained from the following moment equilibrium equation

$$\sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_{sz}^{*}(s_i,z)[x(s_i) - x_s] \sin \phi_i ds_i - \sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_{sz}^{*}(s_i,z)[y(s_i) - y_s] \cos \phi_i ds_i = 0 \tag{41}$$

5.3 Derivation of the Shear Correction Factor

A static shear correction factor is introduced by equating the shear strain energy predicted by the present Timoshenko beam theory, and the shear strain energy obtained from the shearing stress distribution in the cross section

$$\frac{1}{2}\sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_{sz}^{*}(s_i,z)\bar{\gamma}_{sz}^{*}(s_i,z) \sin^2 \phi_i ds_i = \frac{1}{2} V_y(z)\gamma_{yz}(z) \tag{42}$$

After substitution of the expressions for γ_{yz} from constitutive Equations (31), and for $\bar{\gamma}_{sz}^{*}$ obtained from equilibrium in Equation (40), we obtain

$$\frac{1}{2}\sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \frac{1}{\bar{F}_i}(\bar{N}_{sz}^{*}(s_i,z) \sin \phi_i)^2 ds_i = \frac{1}{2}\frac{1}{K_y F_y}(V_y(z))^2 \tag{43}$$

where the beam shear force resultant (V_y) can be expressed as

$$V_y(z) = \sum_{i=1}^{n} \int_{-b_i/2}^{b_i/2} \bar{N}_{sz}^{*} \sin \phi_i ds_i \tag{44}$$

Introducing this expression in Equation (43), we arrive at

$$K_y = \frac{\left[\sum_{i=1}^{n}\left(\sin \phi_i \int_{-b_i/2}^{b_i/2} \bar{N}_{sz}^{*} ds_i\right)\right]^2}{F_y \sum_{i=1}^{n} \frac{1}{\bar{F}_i} \sin^2 \phi_i \int_{-b_i/2}^{b_i/2} (\bar{N}_{sz}^{*})^2 ds_i} \tag{45}$$

An explicit expression for K_y is obtained by replacing the expression for the shear flow Equation (38) in Equation (45) and performing the integrals, yielding

$$K_y = \frac{\left[\sum_{i=1}^{n} b_i \sin \phi_i (S_i^y + c_i^y)\right]^2}{F_y \sum_{i=1}^{n} \frac{b_i}{\bar{F}_i} \sin^2 \phi_i [(S_i^y)^2 + 2c_i^y S_i^y + d_i^y]} \tag{46}$$

where the stiffness parameters of the ith wall, c_i^y and d_i^y, are defined as follows

$$c_i^y = \frac{1}{2} b_i\left[\bar{A}_i\left(\bar{y}_i' - \frac{1}{6} b_i \sin \phi_i\right) + \bar{B}_i \cos \phi_i\right]$$

$$d_i^y = \frac{1}{3} b_i^2\left[(\bar{A}_i)^2\left(\frac{b_i^2}{40} \sin^2 \phi_i - \frac{b_i}{4}\bar{y}_i' \sin \phi_i + (\bar{y}_i')^2\right) + 2\bar{A}_i\bar{B}_i \cos \phi_i\left(\bar{y}_i' - \frac{b_i}{8} \sin \phi_i\right) + (\bar{B}_i)^2 \cos^2 \phi_i\right] \tag{47}$$

The variation of K_y with respect to the geometric dimensions for the case of an I-beam composed of homogeneous walls is shown in Figure 3.

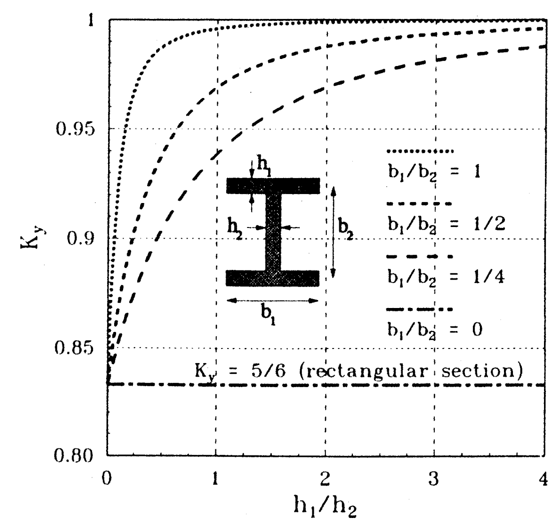

Figure 3. *Variation of the shear correction factor with respect to the geometric dimensions for an homogeneous I-section with a web height to thickness ratio* $b_2/h_2 = 16$.

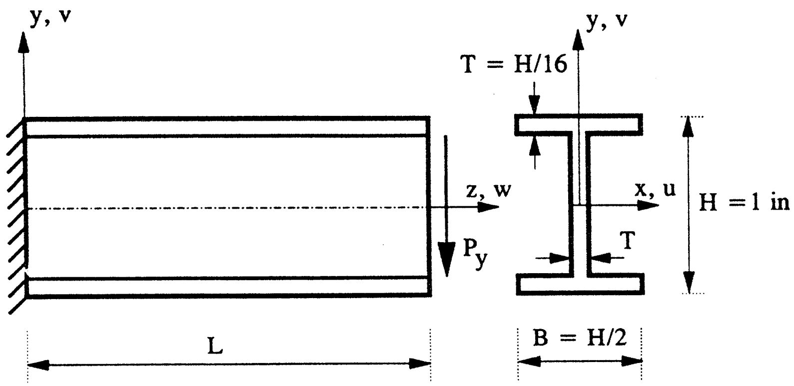

Figure 4. Cantilever I-beam subjected to tip-shear force.

6. NUMERICAL EXAMPLE

A clamped-free I-beam of length L subjected to a tip-shear force $P_y = -100$ lbs is analyzed (see Figure 4). The dimensions of the beam are defined relative to the beam height $H = 1$ in. The material employed is aramid (Kevlar 49)-epoxy with the following elastic constants, $E_1 = 11.02 \times 10^6$ psi (76.0 GPa), $E_2 = 0.80 \times 10^6$ psi (5.50 GPa), $G_{12} = 0.33 \times 10^6$ psi (2.30 GPa), and $\nu_{12} = 0.34$. The lay-up sequence is $[\pm\theta/0]_s$, where the ply angle θ is selected as the design variable. The tip deflection is obtained from the solution of the governing Equation (27) for the specified boundary conditions.

$$v(L) = v_b(L) + v_s(L) = \frac{P_y L^3}{3D_y} + \frac{P_y L}{K_y F_y} \tag{48}$$

The tip deflection is evaluated applying MLB for two different aspect ratios: $L/H = 6$ (Figure 5), and $L/H = 12$ (Figure 6). The results are compared with the values obtained from a refined Finite Element (FE) analysis with ANSYS [26] employing 8-node isoparametric laminated shell elements. The minimum tip deflection for $L/H = 6$ is obtained for $\theta = 16°$, and for $L/H = 12$ is exhibited for $\theta = 9°$. The ratio between the tip shear deflection ($v_s(L)$) and the tip bending deflection ($v_b(L)$) is depicted in Figure 7. The results obtained with MLB (Figure 7) provide insight into the deflection components (bending and shear), that is not available from the FE solution. Ply stresses are computed based on the ply strains obtained from Equation (21) following two different approaches. The first approach considers the stiffness coefficients $\overline{Q}_{ij}$, and the second approach employs the modified stiffness coefficients $\hat{Q}_{ij}$ introduced in Equation (22). Ply axial stresses in the top flange calculated at a distance $z = L/48$ of the fixed end are presented in Figures 8 and 9. Average ply shear stresses in the web evaluated at

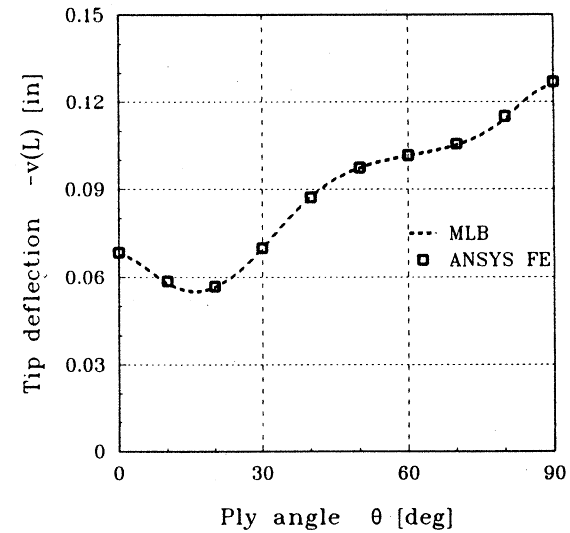

Figure 5. Variation of the tip deflection with respect to the ply angle for a cantilever I-beam with a span to height ratio L/H = 6.

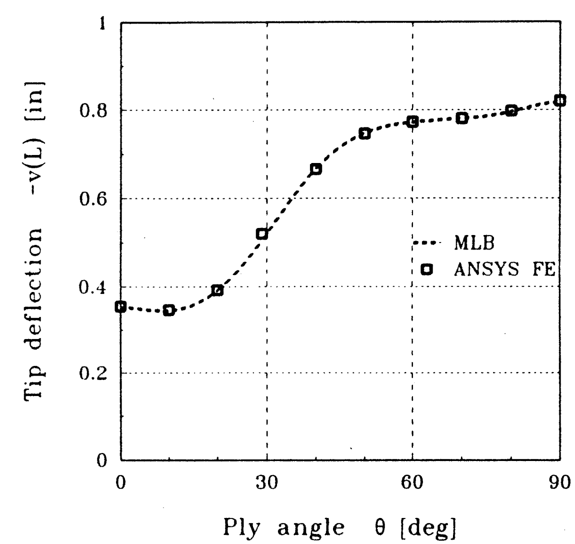

Figure 6. Variation of the tip deflection with respect to the ply angle for a cantilever I-beam with a span to height ratio L/H = 12.

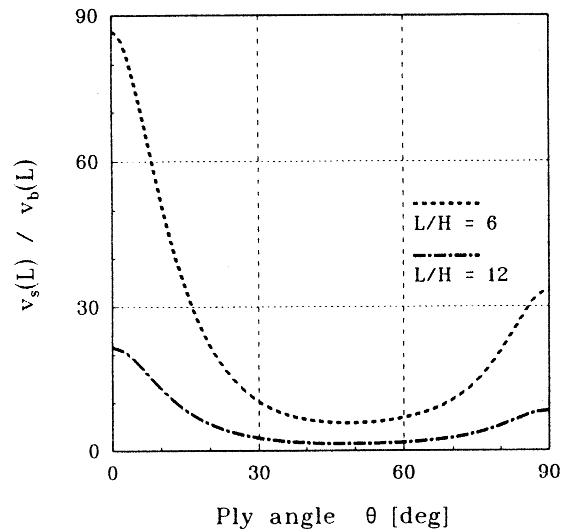

***Figure 7.** Variation of the ratio between the tip shear deflection and the tip bending deflection with respect to the ply angle for a cantilever I-beam.*

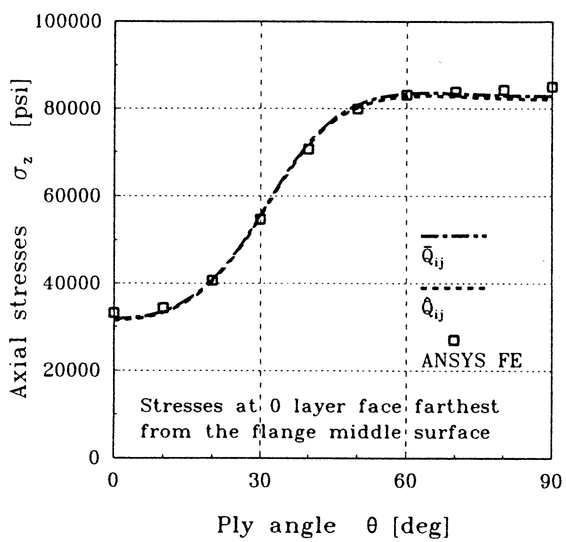

***Figure 8.** Variation of ply axial stresses at a 0 layer in the top flange with respect to the ply angle, measured at z = L/48, for a cantilever I-beam with L/H = 12.*

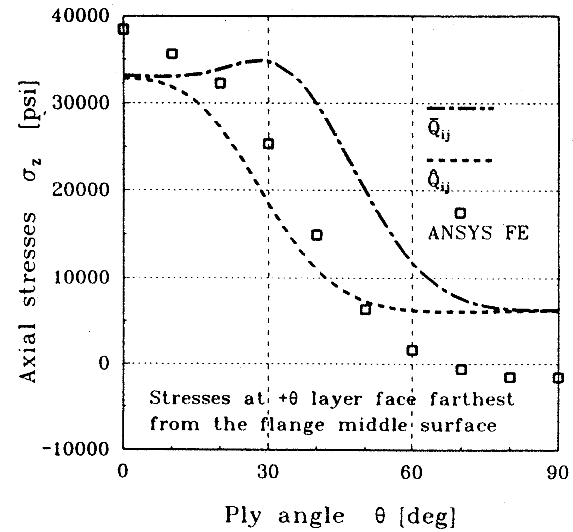

***Figure 9.** Variation of ply axial stresses at a $+\theta$ layer in the top flange with respect to the ply angle, measured at z = L/48, for a cantilever I-beam with L/H = 12.*

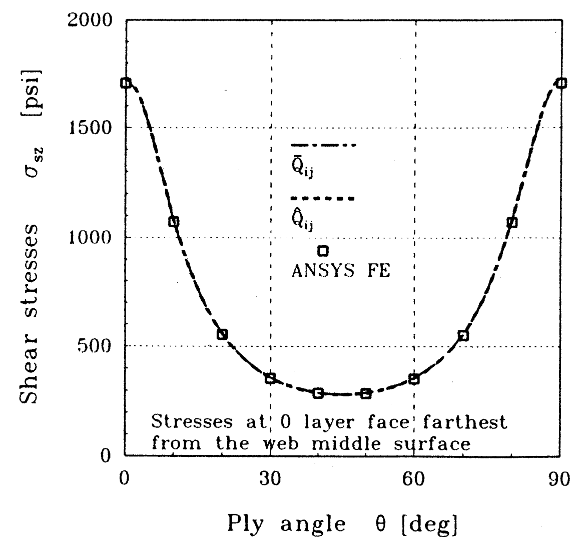

***Figure 10.** Variation of ply average shear stresses at a 0 layer in the web with respect to the ply angle, measured at z = L/2, for a cantilever I-beam with L/H = 12.*

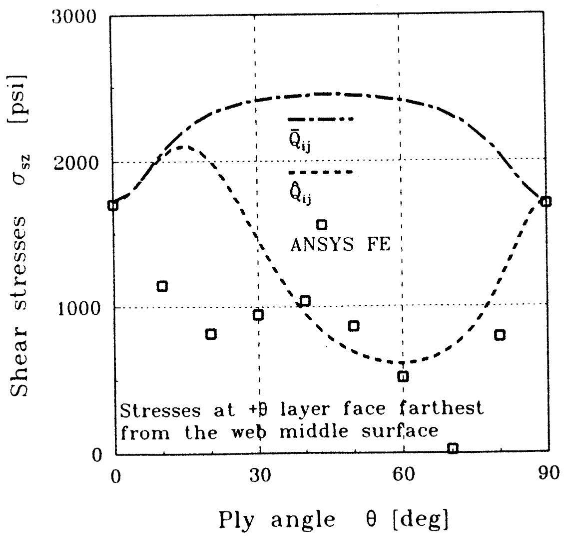

Figure 11. Variation of ply average shear stresses at a $+\theta$ layer in the web with respect to the ply angle, measured at z = L/2, for a cantilever I-beam with L/H = 12.

a distance $z = L/2$ are shown in Figures 10 and 11. The axial and shear stresses at the 0° layer (Figures 8 and 10) obtained by both approaches coincide with the FE results. At the $+\theta$ layer (Figures 9 and 11), the stress computation with the $\hat{Q}_{ij}$ coefficients follows approximately the trend of the FE solution. The difference observed is due to the limitation of a beam theory to model the fully anisotropic response at the lamina level. Nevertheless, we notice that the proposed approach employing the $\hat{Q}_{ij}$ coefficients yields a better approximation to the FE solution than the classical approach with the $\overline{Q}_{ij}$ coefficients.

7. CONCLUSION

The mechanics of laminated beams presented herein intends to bridge a gap between sophisticated existing models and the requirement of a simple but consistent tool for engineering design. For the example presented, the performance of the proposed beam model compared satisfactorily with shell finite elements. In particular, the prediction of deflections, which typically control the design in many civil engineering applications, is remarkably accurate. This approach allows the designer to optimize both the cross-section geometry and the material system for a given objective function. The present formulation could be implemented as a specialized beam finite element, providing a preprocessor to compute the beam stiffness coefficients, and a postprocessor to calculate ply stresses. Further research is advisable in order to evaluate the importance of including warping effects in the design of innovative composite structural shapes. Warping was not included in this work to limit the complexity originated by additional kinematic variables.

8. NOTATION

$[A],[B],[D]$ = laminate stiffness submatrices
$\overline{A}_i,\overline{B}_i,\overline{D}_i,\overline{F}_i$ = stiffness coefficients of the ith wall of a thin-walled laminated beam
A_z,B_y,D_y,F_y = stiffness coefficients of a thin-walled laminated beam
$[a],[d]$ = symmetric laminate compliance submatrices
b_i = width of the ith wall
E_1,E_2,G_{12},ν_{12} = lamina elastic constants
K_y = shear correction factor
L = length of the beam
$\{\overline{N}\} = \{\overline{N}_z,\overline{N}_s,\overline{N}_{sz}\}$ = laminate resultant forces
$\{\overline{M}\} = \{\overline{M}_z,\overline{M}_s,\overline{M}_{sz}\}$ = laminate resultant moments
$N_z(z),M_y(z),V_y(z)$ = beam resultant forces and moment
$\overline{N}^*_{sz}$ = shearing stress distribution (shear flow)
$\overline{N}^o_{sz}$ = uniform closed-cell shear flow
n = number of walls in the cross section
n' = number of walls in a closed cell
$\overline{Q}_{ij}$ = transformed reduced stiffness coefficients
$\hat{Q}_{ij}$ = modified stiffness coefficients for laminated beams
q_y,q_z = transverse and axial applied loads
$S(x_s,y_s)$ = location of the shear center
S^y_i = weighted static moment
(s_i,n_i,z) = local contour coordinate system for the ith wall
$\overline{U}(z)$ = strain energy per unit length
$u(z),v(z)$ = rigid body motions in the x,y directions
$w(z)$ = motion of the neutral axis in the z direction
$\overline{u}(s_i,z),\overline{v}(s_i,z),\overline{w}(s_i,z)$ = motions of the contour in the (s_i,n_i,z) directions
(x,y,z) = Cartesian coordinate system
$x(s_i),y(s_i)$ = parametric functions that define the contour
y' = coordinate with respect to the neutral axis
y_n = coordinate of the neutral axis of bending
y^*_n = coordinate of the neutral axis of combined bending and axial force
$\overline{y}_i$ = centroid coordinate of the ith wall
$\overline{y}'_i$ = centroid coordinate of the ith wall with respect to the neutral axis
$[\alpha],[\beta],[\delta]$ = laminate compliance submatrices
$\{\overline{\epsilon}\} = \{\overline{\epsilon}_z,\overline{\epsilon}_s,\overline{\gamma}_{sz}\}$ = laminate strains
$\{\overline{\kappa}\} = \{\overline{\kappa}_z,\overline{\kappa}_s,\overline{\kappa}_{sz}\}$ = laminate curvatures
$\epsilon^o_z(z),\kappa_y(z),\gamma_{yz}(z)$ = beam strains and curvature
$\overline{\gamma}^*_{sz}$ = shearing strain distribution from equilibrium

ϵ_z, γ_{sz} = ply strains
σ_z, σ_{sz} = ply stresses
Π = total potential energy
ξ = thickness coordinate
ϕ_i = angle between the x axis and the s_i axis
$\psi_y(z)$ = rotation of the cross section

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