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AN ACCURATE DETERMINATION OF STRESSES IN THICK LAMINATES USING A GENERALIZED PLATE THEORY

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SUMMARY

Analytical solutions for displacements and stresses in composite laminates are developed using the laminate plate theory of Reddy. The theory accounts for a desired degree of approximation of the displacements through the laminate thickness, allowing for piecewise approximation of the inplane deformation through individual laminae. The solutions are compared with the 3-D elasticity solutions for the simply supported case and excellent agreement is found. Analytical solutions are also presented for other boundary conditions. The results indicate that the generalized shear deformation plate theory predicts accurate stress distributions in thick composite laminates.

INTRODUCTION

Among the displacement-based refined theories that are available in the literature the first one is due to Basset.[1] Basset assumed that the three displacement components in a shell can be expanded as a linear combination of the thickness co-ordinate and unknown functions of position in the reference surface of the plate. The Basset type displacement expansions were used by Hildebrand *et al.*,[2] Hencky,[3] Mindlin[4] and recently by Reddy[5–7] to develop various first-order and higher-order plate theories. An *n*th-order theory is one in which the displacements (often, the inplane displacements) are expanded in terms of the thickness co-ordinate up to the *n*th power. The equations of equilibrium or motion are derived often using the principle of virtual displacements. Most of the refined theories do not require vanishing of the transverse shear stresses on the bounding planes of the plate. The third-order theory advanced by Reddy[5–7] satisfies the traction free boundary conditions on the top and bottom faces of a laminate composite of orthotropic layers.

All laminate plate theories derived from the Basset type expansion assume that the displacements vary through the thickness of the laminate according to a single expression, not allowing for possible discontinuities in the slopes of the deflections at the interfaces of two individual laminae. Recently, Reddy[8] presented a laminate plate theory that allows piecewise representation of

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displacements through individual laminae of a laminated plate. The theory was extended to include the von Kármán non-linearity by Reddy and Barbero.[9] Similar, but different, theories have appeared in the literature (see References 10–13). The present study deals with the analytical solutions to the general laminate theory presented in Reference 8, and an evaluation of the accuracy of the stresses predicted by the theory in light of the 3-D elasticity results. The development of analytical solutions to the layer-wise displacement theory is by no means simple, especially for boundary conditions other than simply supported. Although the basic theory was presented in Reference 8, its solutions, hence the accuracy, have not been investigated previously. The present work is in the same spirit as the works of Pagano,[14, 15] who presented analytical solutions of the well-known first-order shear deformation theory to investigate shear deformation effects in composite laminates.

THEORETICAL FORMULATION

Consider a laminated plate composed of N orthotropic laminae, each being oriented arbitrarily with respect to the laminate (x, y) co-ordinates, which are taken to be in the midplane of the laminate. The displacements (u_1, u_2, u_3) at a generic point (x, y, z) in the laminate are assumed to be of the form (see Reference 8),

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + U(x, y, z) \\ u_2(x, y, z) &= v(x, y) + V(x, y, z) \\ u_3(x, y, z) &= w(x, y) \end{aligned} \tag{1}$$

where (u, v, w) are the displacements of a point $(x, y, 0)$ on the reference plane of the laminate, and U and V are functions which vanish on the reference plane:

$$U(x, y, 0) = V(x, y, 0) = 0 \tag{2}$$

Although the displacement component u_3 is assumed to be constant through laminate thickness in the present study, it is not a restriction of the GLPT as developed in Reference 8. The constant state of u_3 through the thickness is justified in view of the relatively small magnitudes of the transverse normal stress compared to the other stress components, and the assumption is used extensively in most refined plate theories.

In order to reduce the three-dimensional theory to a two-dimensional one, Reddy[8] suggested that the out-of-plane displacement functions be expanded as a linear combination of undetermined functions of (x, y) and known functions of z:

$$\begin{aligned} U(x, y, z) &= \sum_{j=1}^{n} U^j(x, y)\Phi_j(z) \\ V(x, y, z) &= \sum_{j=1}^{n} V^j(x, y)\Phi_j(z) \end{aligned} \tag{3}$$

where U^j and V^j are undetermined coefficients and Φ_j are any continuous functions that satisfy the condition

$$\Phi_j(0) = 0 \quad \text{for all } j = 1, 2, \ldots, n \tag{4}$$

The approximation in equation (3) can also be viewed as the global semi-discrete finite-element approximations (see Reference 16) of U and V through the thickness. In that case Φ_j denote the global interpolation functions, and U^j and V^j are the global nodal values of U and V (and possibly their derivatives) at the nodes through the thickness of the laminate. For example, a finite-element approximation based on the Lagrangian interpolation through the thickness can be obtained from equation (3) by setting $n = pN + 1$, where

N = number of layers through thickness

p = degree of the global interpolation polynomials, $\Phi_j(z)$, and

U^j, V^j = global nodal values of U and V.

For $p = 1$ (i.e. linear interpolation), we have

$$\begin{aligned} U^1 &= u_1^{(1)},\ U^2 = u_2^{(1)} = u_1^{(2)}, \ldots,\ U^k = u_2^{(k-1)} = u_1^{(k)} \\ V^1 &= v_1^{(1)},\ V^2 = v_2^{(1)} = v_1^{(2)}, \ldots,\ V^k = v_2^{(k-1)} = v_1^{(k)} \end{aligned} \tag{5}$$

where $u_j^{(k)}$, for example, denotes the value of U at the jth node of the kth lamina. The linear global interpolation functions are given by

$$\Phi_k(z) = \begin{matrix} \psi_2^{(k-1)}(z), & z_{k-1} \leqslant z \leqslant z_k \\ \psi_1^{(k)}(z), & z_k \leqslant z \leqslant z_{k+1} \end{matrix} \quad (k = 1, 2, \ldots, N) \tag{6}$$

where $\psi_j^{(k)}$ $(j = 1, 2)$ is the local (i.e. layer) Lagrange interpolation function associated with the jth node of the kth layer. If the mid-plane does not coincide with an interface, it is used as an interface to satisfy equation (2). If U^j, V^j correspond to the midplane interface, equation (2) is satisfied by setting U^j and $V^j = 0$. Therefore n reduces to $n = N$.

The equilibrium equations of the theory can be derived using the principle of virtual displacements (see Reference 8):

$$\begin{aligned} &N_{x,x} + N_{xy,y} = 0, \quad N_{xy,x} + N_{y,y} = 0 \\ &Q_{x,x} + Q_{y,y} + f = 0 \\ &N^j_{x,x} + N^j_{xy,y} - Q^j_x = 0, \quad N^j_{xy,x} + N^j_{y,y} - Q^j_y = 0, \quad (j = 1, 2, \ldots, n) \end{aligned} \tag{7}$$

where

$$\begin{aligned} (N_x, N_y, N_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\,\mathrm{d}z \\ (Q_x, Q_y) &= \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz})\,\mathrm{d}z \\ (N^j_x, N^j_y, N^j_{xy}) &= \int_{-h/2}^{h/2} (\sigma_x, \sigma_y, \sigma_{xy})\Phi_j(z)\,\mathrm{d}z \\ (Q^j_x, Q^j_y) &= \int_{-h/2}^{h/2} (\sigma_{xz}, \sigma_{yz})\frac{\mathrm{d}\Phi_j}{\mathrm{d}z}(z)\,\mathrm{d}z \end{aligned} \tag{8}$$

$(\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{xz}, \sigma_{yz})$ are the stresses and f is the distributed transverse load.

There are $2n + 1$ differential equations in $(2n + 1)$ variables (u, v, w, U^j, V^j). The form of the geometric and force boundary conditions is given below:

Geometric (Essential)	Force (Natural)	
u	$N_x n_x + N_{xy} + n_y$	
v	$N_{xy} n_x + N_y n_y$	
w	$Q_x n_x + Q_y n_y$	(9)
U^j	$N^j_x n_x + N^j_{xy} n_y$	
V^j	$N^j_{xy} n_x + N^j_y n_y$	

where (n_x, n_y) denote the direction cosines of a unit normal to the boundary of the midplane Ω.

The constitutive equations of the laminate are given by

$$
\begin{Bmatrix} N_x \\ N_y \\ N_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 \\ 0 & 0 & 0 & A_{55} & A_{45} \\ 0 & 0 & 0 & A_{45} & A_{44} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} + \sum_{j=1}^{n} \begin{bmatrix} B_{11}^j & B_{12}^j & B_{16}^j & 0 & 0 \\ B_{12}^j & B_{22}^j & B_{26}^j & 0 & 0 \\ B_{16}^j & B_{26}^j & B_{66}^j & 0 & 0 \\ 0 & 0 & 0 & \bar{B}_{55}^j & \bar{B}_{45}^j \\ 0 & 0 & 0 & \bar{B}_{45}^j & \bar{B}_{44}^j \end{bmatrix} \begin{Bmatrix} \dfrac{\partial U^j}{\partial x} \\ \dfrac{\partial V^j}{\partial y} \\ \dfrac{\partial U^j}{\partial y} + \dfrac{\partial V^j}{\partial x} \\ U^j \\ V^j \end{Bmatrix}
$$

$$
\begin{Bmatrix} N_x^j \\ N_y^j \\ N_{xy}^j \\ Q_x^j \\ Q_y^j \end{Bmatrix} = \begin{bmatrix} B_{11}^j & B_{12}^j & B_{16}^j & 0 & 0 \\ B_{12}^j & B_{22}^j & B_{26}^j & 0 & 0 \\ B_{16}^j & B_{26}^j & B_{66}^j & 0 & 0 \\ 0 & 0 & 0 & \bar{B}_{55}^j & \bar{B}_{45}^j \\ 0 & 0 & 0 & \bar{B}_{45}^j & \bar{B}_{44}^j \end{bmatrix} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial v}{\partial y} \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial y}{\partial x} \\ \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} + \sum_{k=1}^{n} \begin{bmatrix} D_{11}^{jk} & D_{12}^{jk} & D_{16}^{jk} & 0 & 0 \\ D_{12}^{jk} & D_{22}^{jk} & D_{26}^{jk} & 0 & 0 \\ D_{16}^{jk} & D_{26}^{jk} & D_{66}^{jk} & 0 & 0 \\ 0 & 0 & 0 & \bar{D}_{55}^{jk} & \bar{D}_{45}^{jk} \\ 0 & 0 & 0 & \bar{D}_{45}^{jk} & \bar{D}_{44}^{jk} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial U^k}{\partial x} \\ \dfrac{\partial V^k}{\partial y} \\ \dfrac{\partial U^k}{\partial y} + \dfrac{\partial V^k}{\partial x} \\ U^k \\ V^k \end{Bmatrix} \qquad (10a)
$$

where

$$A_{pq} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{pq}^{(k)} \, dz \quad (p, q = 1, 2, 6; 4, 5)$$

$$B_{pq}^{j} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{pq}^{(k)} \Phi^j \, dz \quad (p, q = 1, 2, 6)$$

$$D_{pq}^{ji} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{pq}^{(k)} \Phi^j \Phi^i \, dz \quad (p, q = 1, 2, 6) \tag{10b}$$

$$\bar{B}_{pq}^{j} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{pq}^{(k)} \frac{d\Phi^j}{dz} dz \quad (p, q = 4, 5)$$

$$\bar{D}_{pq}^{ji} = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} Q_{pq}^{(k)} \frac{d\Phi^j}{dz} \frac{d\Phi^i}{dz} dz \quad (p, q = 4, 5)$$

for all $i, j = 1, 2, \ldots, n$.

ANALYTICAL SOLUTIONS

In this study we use a finite-element approximation based on the linear Lagrangian interpolation through the thickness. In order to satisfy the conditions (2), we choose the midplane as an interface and set $U(x, y, 0) = V(x, y, 0) = 0$. A convenient way to accomplish this is to eliminate the variables U^j and V^j at the midplane; therefore the number of necessary terms in (3) reduces to $n = N$, the number of layers.

The coefficients A_{pq} have the same meaning as in the classical plate theory (CPT). The calculation of the coefficients B_{pq}^j involves only the properties of the layers adjacent to the jth interface because the functions Φ_j are identically zero at other interfaces. The same is true for the coefficients D_{pq}^{ji}.

Since the approximation through the thickness is built with a finite-element family of functions, a standard, one-dimensional finite-element procedure can be used to perform the integration. This makes the procedure very general with respect to the number of layers, thicknesses and properties that can be handled. The contribution of each layer to its adjacent nodes (located on the interfaces) is then assembled in the usual way (see Reddy[16]). The $[B^j]$ array has an entry for each interface. The array $[D^{ji}]$ is equivalent to the mass matrix, and has a half bandwidth of 2.

Here we consider analytical solutions for the case of cylindrical bending of a plate strip under various boundary conditions and for simply supported cross-ply plates.

Cylindrical bending

The plate equations (7) can be specialized to cylindrical bending by taking $v = 0$, $V^j = 0$, $u = u(x)$, $U^j = U^j(x)$, $w = w(x)$. The equivalent equations can be written as

$$A_{11} \frac{du^2}{dx^2} + \sum_{k=1}^{N} B_{11}^k \frac{d^2 U^k}{dx^2} = 0 \qquad A_{55} \frac{d^2 w}{dx^2} + \sum_{k}^{N} B_{55}^k \frac{dU^k}{dx} + f = 0$$
$$B_{11}^j \frac{d^2 u}{dx} - B_{55}^j \frac{dw}{dx} + \sum_{k=1}^{N} \left[D_{11}^{jk} \frac{d^2 U^k}{dx^2} - D_{55}^{jk} U^k \right] = 0 \tag{11}$$

Equations (11) consist of $N + 2$ equations in $u, w, U^1, U^2, \ldots, U^N$ unknowns, where N denotes the number of layers.

We consider the case of $N = 2$ to illustrate the solution by the state-space procedure.[17] First we transform the system of equations (11) to a system of first-order ordinary differential equations.

Introducing the unknowns x_i through the relations

$$\begin{array}{llll} \alpha_1 = u & \alpha_3 = w & \alpha_5 = U^1 & \alpha_7 = U^2 \\ \alpha_2 = u' & \alpha_4 = w' & \alpha_6 = (U^1)' & \alpha_8 = (U^2)' \end{array} \tag{12}$$

we obtain a system of ordinary differential equations from equation (11),

$$\{\alpha'\} = [\bar{A}]\{\alpha\} + \{F\} \tag{13a}$$

where

$$\begin{aligned} [\bar{A}] &= [A]^{-1}[B] \\ \{F\} &= [A]^{-1}\{f\} \end{aligned} \tag{13b}$$

$$[A] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & A_{11} & 0 & 0 & 0 & B_{11}^1 & 0 & B_{11}^2 \\ 0 & 0 & 0 & A_{55} & 0 & 0 & 0 & 0 \\ 0 & B_{11}^1 & 0 & 0 & 0 & D_{11}^{11} & 0 & D_{11}^{12} \\ 0 & B_{11}^2 & 0 & 0 & 0 & D_{11}^{21} & 0 & D_{11}^{22} \end{bmatrix}$$

$$[B] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -B_{55}^1 & 0 & -B_{55}^2 \\ 0 & 0 & 0 & B_{55}^1 & D_{55}^{11} & 0 & D_{55}^{12} & 0 \\ 0 & 0 & 0 & B_{55}^2 & D_{55}^{21} & 0 & D_{55}^{22} & 0 \end{bmatrix}; \quad \{f\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -f(x) \\ 0 \\ 0 \end{Bmatrix} \tag{13c}$$

As a particular example, we consider a plate strip made of an isotropic material ($E = 30 \times 10^6$ psi, $E/G = 2{\cdot}5$, $h = 2$ in) in cylindrical bending. A uniformly distributed transverse load of intensity f_0 is used. For this case $[\bar{A}]$ becomes

$$[\bar{A}] = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6/5 & 0 & -6/5 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -6/5 & 3 & 0 & 9/5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 6/5 & 9/5 & 0 & 3 & 0 \end{bmatrix}; \quad \{F\} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -f_0/24 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{14}$$

The eigenvalues of the matrix $[\bar{A}]$ are

$$\begin{aligned} \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = 0 \\ \lambda_7 \cong 2{\cdot}19 \\ \lambda_8 = -\lambda_7 \end{aligned} \tag{15}$$

For this case we have only four linearly independent eigenvectors. For the eigenvalue $\lambda = 0$ the eigenvector is of the form

$$\{\xi_1\} = \{k_1, 0, k_2, 0, 0, 0, 0, 0\}^{\mathrm{T}}$$

To obtain other linearly independent solutions, we use the solution procedure presented by Goldbery and Schwartz.[17] First we set

$$(\bar{A} - \lambda I)\{\xi_2\} = \{\xi_1\}$$

and find that $\lambda = 0$, and therefore

$$[\bar{A}]\{\xi_2\} = \{\xi_1\}$$

This yields

$$\{\xi_2\} = \{k_3, k_1, k_4, k_2, k_2, 0, -k_2, 0\}^{\mathrm{T}}$$

Next we set $[\bar{A}]\{\xi_3\} = \{\xi_2\}$ and find

$$\{\xi_3\} = \{k_5, k_3, k_6, k_4, k_4, k_2, -k_4, -k_2\}^{\mathrm{T}} \text{ and } k_1 = 0$$

which annihilates one of the eigenvectors.

Repeating the procedure, we obtain

$$\{\xi_4\} = \{k_7, k_5, k_8, k_6, \tfrac{5}{6}(k_2 + \tfrac{6}{5}k_6), k_4, -\tfrac{5}{6}(k_2 + \tfrac{6}{5}k_6), -k_4\}^{\mathrm{T}}, \quad k_3 = 0$$

$$\{\xi_5\} = \{k_7, k_5, k_8, k_6, -\tfrac{4}{51}(5k_2 + 6k_6), k_4, \tfrac{4}{51}(5k_2 + 6k_6), -k_4\}^{\mathrm{T}}$$

Lastly, we set $[\bar{A}]\{\xi_6\} = \{\xi_5\}$, and arrive at the condition $k_2 = 0$, which annihilates the only eigenvector left, so the process is terminated. The particular solution of the problem is

$$\{\alpha_{\mathrm{p}}(x)\} = \int_0^x [\phi(x-s)\cdot\phi^{-1}(0)]\cdot\{F(s)\}\,\mathrm{d}s \tag{16}$$

where

$$\phi(x) = \begin{bmatrix} 0 & 1 & 0 & x & 0 & 0 & & \\ 0 & 0 & 0 & 1 & 0 & 0 & & \\ 1 & 0 & x & 0 & x^2/2 & x^3/6 & & \\ 0 & 0 & 1 & 0 & x & x^2/2 & & \\ 0 & 0 & 1 & 0 & x & 5/6 + x^2/2 & \{\xi_7\}\mathrm{e}^{\lambda_7 x} & \{\xi_8\}\mathrm{e}^{\lambda_8 x} \\ 0 & 0 & 0 & 0 & 1 & x & & \\ 0 & 0 & -1 & 0 & -x & -5/6 - x^2/2 & & \\ 0 & 0 & 0 & 0 & -1 & -x & & \end{bmatrix} \tag{17}$$

The general solution is given by

$$\{\alpha(x)\} = \phi(x)\cdot\{k\} + \{\alpha_{\mathrm{p}}(x)\} \tag{18}$$

where $\{k\}$ is the vector of constants, which can be found using the boundary conditions. For example, for a clamped–clamped case the boundary conditions at $x=\pm a/2$ are

$$
\begin{aligned}
u(-a/2) &= u(a/2) = 0\\
w(-a/2) &= w(a/2) = 0\\
U^1(-a/2) &= U^1(a/2) = 0\\
U^2(-a/2) &= U^2(a/2) = 0
\end{aligned}
$$

which give us eight equations to compute the eight constants in the vector $\{k\}$.

For the particular choice of $a=20$ in and uniformly distributed load of intensity $f_0 = 1$ lb/in, the solution is given by

$$
\begin{aligned}
u(x) &= (-5{\cdot}94\times10^{-29}\mathrm{e}^{-2{\cdot}19x} - 3{\cdot}07\times10^{-26}\mathrm{e}^{2{\cdot}19x})\cdot10^{-6}\\
w(x) &= \left(\frac{x^2}{48}-\frac{x^4}{480}+\frac{5}{12}x^2-22{\cdot}9167\right)\cdot10^{-6}\\
U^1(x) &= \left(\frac{5}{6}x-\frac{x^3}{120}+1{\cdot}18\times10^{-28}\mathrm{e}^{-2{\cdot}19x}+6{\cdot}15\times10^{-26}\mathrm{e}^{2{\cdot}19x}\right)\cdot10^{-6}\\
U^2(x) &= \left(-\frac{5}{6}x+\frac{x^3}{120}+1{\cdot}18\times10^{-28}\mathrm{e}^{-2{\cdot}19x}+6{\cdot}15\times10^{-26}\mathrm{e}^{2{\cdot}19x}\right)\cdot10^{-6}
\end{aligned}
\tag{19}
$$

Plots of the transverse deflection w as a function of the thickness ratio a/h are shown in Figure 1 for three types of boundary conditions: cantilever, simply supported and clamped at both ends. For all cases a uniformly distributed load is used. Values for the exact 3-D solution[14] for the simply supported case are also shown. The deflections are normalized with respect to the CPT solution. The present solution is in excellent agreement with the 3-D elasticity solution. We note that the clamped plate exhibits more shear deformation.

Similar results are presented in Figure 2 for a two-layer cross-ply [0°/90] plate strip. The material properties of a ply are taken to be those of a graphite–epoxy material:

$$
\begin{aligned}
E_1 &= 19{\cdot}2\times10^6\ \text{psi}\\
E_2 &= 1{\cdot}56\times10^6\ \text{psi}\\
G_{12} &= G_{13} = 0{\cdot}82\times10^6\ \text{psi}\\
G_{23} &= 0.523\times10^6\ \text{psi}\\
\nu_{12} &= \nu_{13} = 0{\cdot}24\\
\nu_{23} &= 0{\cdot}49
\end{aligned}
\tag{20}
$$

Once again, it is clear that the present theory yields very accurate results.

Simply supported plates

Consider a rectangular ($a\times b$) cross-ply laminate, not necessarily symmetric, composed of N layers. For such a plate the laminate constitutive equations (10) simplify because $A_{16}=A_{26}=A_{45}$

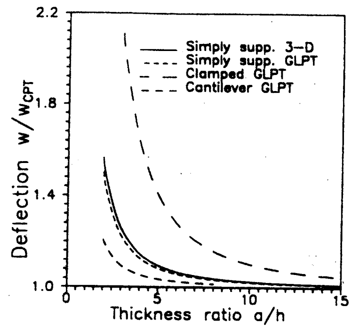

Figure 1. Normalized maximum deflection versus side to thickness ratio for an isotropic plate strip under uniform transverse load

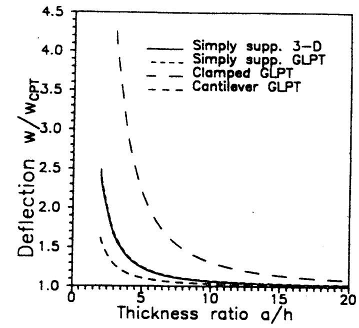

Figure 2. Normalized maximum deflection versus side to thickness ratio for two-layer cross-ply plate strip under uniform transverse load

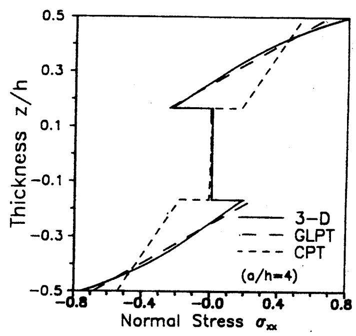

Figure 3. Variation of the axial stress through the thickness of three-layer cross-ply (0/90/0) laminate under sinusoidal varying transverse load

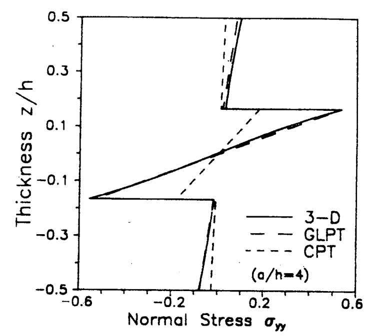

Figure 4. Variation of the axial stress through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

$= B_{16}^k = B_{26}^k = B_{45}^k = D_{16}^{jk} = D_{26}^{jk} = D_{45}^{jk} = 0$. The governing equations become

$$
\begin{aligned}
&A_{11} u_{,xx} + A_{12} v_{,yx} + A_{66}(u_{,yy} + v_{,xy}) \\
&\qquad + \sum_{k=1}^{N} [B_{11}^k U_{,xx}^k + B_{12}^k V_{,yx}^k + B_{66}^k (U_{,yy}^k + V_{,xy}^k)] = 0 \\
&A_{66}(u_{,yx} + v_{,xx}) + A_{12} u_{,xy} + A_{22} v_{,yy} \\
&\qquad + \sum_{k=1}^{N} [B_{66}^k (U_{,yx}^k + V_{,xx}^k) + B_{12}^k U_{,xy}^k + B_{22}^k V_{,yy}^k] = 0 \\
&A_{55} w_{,xx} + A_{44} w_{,yy} + \sum_{k=1}^{N} [B_{55}^k U_{,x}^k + B_{44}^k V_{,y}^k] + f = I_0 \ddot{w} \\
&B_{11}^j u_{,xx} + B_{12}^j v_{,yx} + B_{66}^j (v_{,yy} + v_{,xy}) - B_{55}^j w_{,x} \\
&\qquad + \sum_{k=1}^{N} [D_{11}^{jk} U_{,xx}^k + D_{12}^{jk} V_{,yx}^k + D_{66}^{jk} (U_{,yy}^k + V_{,xy}^k) - D_{55}^{jk} U^k] = 0 \\
&B_{66}^j (u_{,yx} + v_{,xx}) + B_{12}^j u_{,xy} + B_{22}^j v_{,yy} - B_{44}^j w_{,y} \\
&\qquad + \sum_{k=1}^{N} [D_{66}^{jk} (U_{,yx}^k + V_{,xx}^k) + D_{12}^{jk} U_{,xy}^k + D_{22}^{jk} V_{,yy}^k - D_{44}^{jk} V^k] = 0
\end{aligned}
\tag{21}
$$

for $i, j = 1, 2, \ldots, N$. These equations are subject to the boundary conditions

$$\begin{aligned} &v = w = V^k = N_x = N_x^k = 0; \quad x = 0, a; \quad k = 1, \ldots, N \\ &u = w = U^k = N_y = N_y^k = 0; \quad y = 0, b; \quad k = 1, \ldots, N \end{aligned} \tag{22}$$

These boundary conditions are identically satisfied by the following expressions for displacements (i.e Navier's solution procedure is used):

$$\begin{aligned} u &= \sum_{m,n}^{\infty} X_{mn} \cos \alpha x \sin \beta y \\ v &= \sum_{m,n}^{\infty} Y_{mn} \sin \alpha x \cos \beta y \\ w &= \sum_{m,n}^{\infty} W_{mn} \sin \alpha x \sin \beta y \\ U^k &= \sum_{m,n}^{\infty} R_{mn}^k \cos \alpha x \sin \beta y \\ V^k &= \sum_{m,n}^{\infty} S_{mn}^k \sin \alpha x \cos \beta y \end{aligned} \tag{23a}$$

where

$$\alpha = \frac{m\pi}{a}; \quad \beta = \frac{n\pi}{b}; \quad k = 1, \ldots, N$$

The transverse load can be expanded in double Fourier series

$$f(x, y) = \sum_{m,n}^{\infty} q_{mn} \sin \alpha x \sin \beta y \tag{23b}$$

Substitution of these expressions into the governing equations gives a system of $2N + 3$ equations for each of the Fourier modes (m, n), from which we obtain the coefficients $(X_{mn}, Y_{mn}, W_{mn}, R_{mn}^k, S_{mn}^k)$:

$$\begin{bmatrix} [k] & [k^j] \\ [k^j]^{\mathrm{T}} & [k^{jk}] \end{bmatrix} \begin{Bmatrix} \{\Delta^1\} \\ \{\Delta^2\} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ q_{mn} \\ 0 \end{Bmatrix} \tag{24}$$

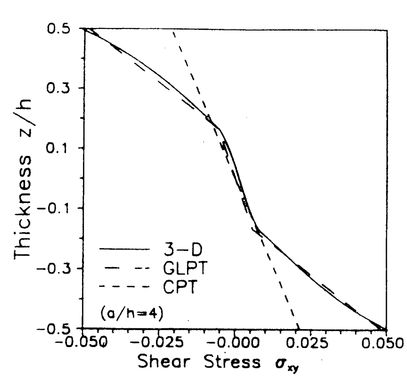

Figure 5. Variation of the shear stress σ_{xy} through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

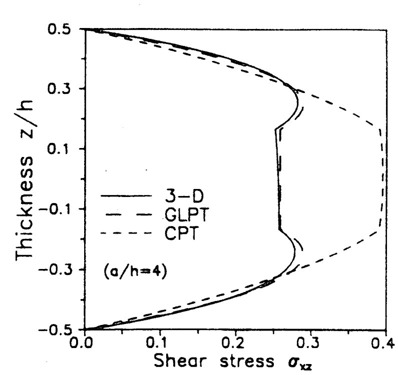

Figure 6. Variation of the transverse shear stress through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

where $\{\Delta^1\}^{\mathrm{T}} = \{X_{mn}, Y_{mn}, W_{mn}\}$, $\{\Delta^2\}^{\mathrm{T}} = \{R^k_{mn}, S^k_{mn}\}$, and the coefficients $[k]$, $[k^{jk}]$ and $[k^j]$ are given in the Appendix.

Once the coefficients $(X_{mn}, Y_{mn}, W_{mn}, R^k_{mn}, S^k_{mn}\}$ are obtained, the inplane stresses can be computed from the constitutive equations as

$$\begin{aligned}
\sigma_x(x, y, z) = &- \sum_{m,n}^{\infty}\left\{\left[Q_{11}\alpha\left(X_{mn} + \sum_{k=1}^{N} R^k_{mn}\Phi^k(z)\right)\right.\right.\\
&\left.\left. + Q_{12}\beta\left(Y_{mn} + \sum_{k=1}^{N} S^k_{mn}\Phi^k(z)\right)\right]\sin\alpha x \sin\beta y\right\}\\
\sigma_y(x, y, z) = &- \sum_{m,n}^{\infty}\left\{\left[Q_{12}\alpha\left(X_{mn} + \sum_{k=1}^{N} R^k_{mn}\Phi^k(z)\right)\right.\right.\\
&\left.\left. + Q_{22}\beta\left(Y_{mn} + \sum_{k=1}^{N} S^k_{mn}\Phi^k(z)\right)\right]\sin\alpha x \sin\beta y\right\}\\
\sigma_{xy}(x, y, z) = &\, Q_{66}\sum_{m,n}^{\infty}\left\{\left[\beta\left(X_{mn} + \sum_{k=1}^{N} R^k_{mn}\Phi^k(z)\right)\right.\right.\\
&\left.\left. + \alpha\left(Y_{mn} + \sum_{k=1}^{N} S^k_{mn}\Phi^k(z)\right)\right]\cos\alpha x \cos\beta y\right\}
\end{aligned} \tag{25}$$

The shear stresses are computed using the equilibrium equations of the 3-D elasticity and enforcing continuity of stresses along the interfaces:

$$\begin{aligned}
\sigma_{xz}(x, y, z) = &\sum_{m,n}^{\infty}[\{[(Q_{11}\alpha^2 + Q_{66}\beta^2)X_{mn}\\
&+ (Q_{12} + Q_{66})\alpha\beta Y_{mn}]z + \sum_{k=1}^{N}\{[(Q_{11}\alpha^2 + Q_{66}\beta^2)R^k_{mn}\\
&+ (Q_{12} + Q_{66})\alpha\beta S^k_{mn}]\textstyle\int\Phi^k \mathrm{d}z\} + H_{\mathrm{i}}\}\cos\alpha x \sin\beta y]\\
\sigma_{yz}(x, y, z) = &\sum_{m,n}^{\infty}[\{[(Q_{66} + Q_{12})\alpha\beta X_{mn} + (Q_{66}\alpha^2 + Q_{22}\beta^2)Y_{mn}]z\\
&+ \sum_{k=1}^{N}\{[(Q_{66} + Q_{12})\alpha\beta R^k_{mn} + (Q_{66}\alpha^2 + Q_{22}\beta^2)S^k_{mn}]\textstyle\int\phi^k\,\mathrm{d}z\}\\
&+ G_{\mathrm{i}}\}\sin\alpha x \cos\beta y]
\end{aligned} \tag{26}$$

where H_{i}, G_{i} are constants introduced to satisfy the continuity of stresses.

To assess the quality of the theory we consider a three-ply symmetric laminate, simply supported, and subjected to sinusoidal transverse load. This problem has the 3-D elasticity solution[15] and the classical plate theory (CPT) solution. The high quality of the solutions obtained with this theory can be fully appreciated considering the stress distributions through the thickness for $\sigma_x, \sigma_y, \sigma_{xy}, \sigma_{yz}$ and σ_{xz} for $a/h=4$ (see Figures 3–7), and $a/h=10$ (see Figures 8–12). The material properties of each ply are

$$\begin{aligned}
&E_1/E_2 = 25{\cdot}0, \quad G_{12} = 0{\cdot}5E_2, \quad G_{13} = G_{12}, \quad G_{23} = 0{\cdot}2E_2\\
&\nu_{12} = \nu_{13} = 0{\cdot}25
\end{aligned} \tag{27}$$

All stresses are non-dimensionalized with respect to the applied load.

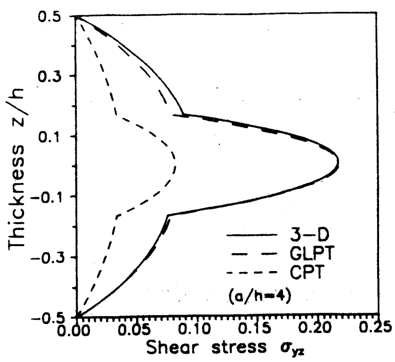

Figure 7. Variation of transverse shear stress σ_{yz} through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

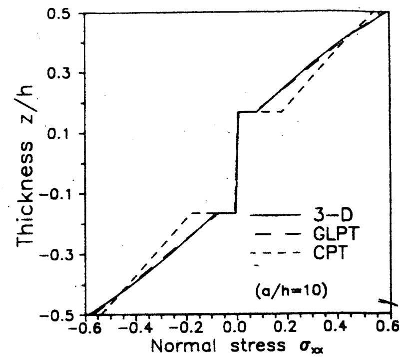

Figure 8. Variation of the normal stress σ_{xx} through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

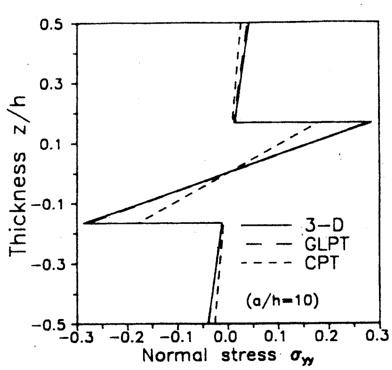

Figure 9. Variation of the normal stress σ_{yy} through the thickness of a three-layer cross-ply (0/90/0) laminate under sinusoidal transverse load

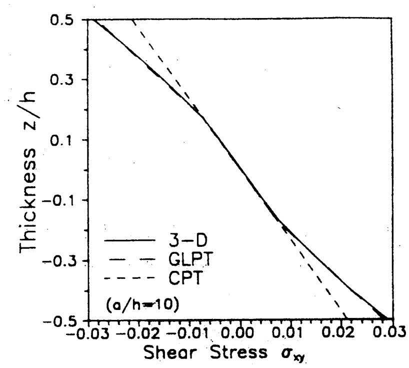

Figure 10. Variation of the shear stress σ_{xy} through the thickness of a three-layer cross-ply laminate under sinusoidal transverse load

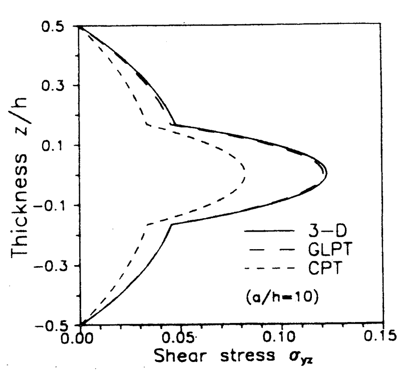

Figure 11. Variation of the transverse shear stress σ_{yz} through the thickness of a three-layer cross-ply laminate under sinusoidal transverse load

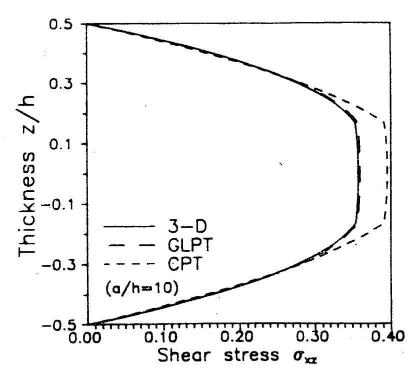

Figure 12. Variation of the transverse shear stress σ_{xz} through the thickness of a three-layer cross-ply laminate under sinusoidal transverse load

The deflection $w(x, y)$ obtained in the present theory coincides with the exact 3-D solution and is not shown here. In all cases the present solutions for stresses are in excellent agreement with the 3-D elasticity solutions, whereas the CPT solutions are considerably in error.

CONCLUSIONS

The analytical solutions of the generalized laminate plate theory are presented, and its accuracy is investigated by comparison with the 3-D elasticity theory. The agreement is found to be excellent, even for very thick plates. The theory gives accurate interlaminar stress distributions, and should prove to be very useful in the failure analysis of composite laminates. The theory can be used to investigate vibration, stability and transient response of composite laminates, and extension of the theory to study delaminations is currently underway.

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APPENDIX

Coefficients of the matrices in equation (24)

$$k_{11} = -A_{11}\alpha^2 - A_{66}\beta^2$$
$$k_{12} = -(A_{12} + A_{66})\alpha\beta; \quad k_{21} = k_{12}$$
$$k_{22} = -A_{22}\beta^2 - A_{66}\alpha^2; \quad k_{13} = k_{31} = 0$$
$$k_{33} = -A_{44}\beta^2 - A_{55}\alpha^2; \quad k_{23} = k_{32} = 0$$
$$k^j_{11} = -B^j_{11}\alpha^2 - B^j_{66}\beta^2$$
$$k^j_{12} = -(B^j_{12} + B^j_{66})\alpha\beta; \quad k^j_{21} = k^j_{12}$$
$$k^j_{22} = -B^j_{22}\beta^2 - B^j_{66}\alpha^2$$
$$k^j_{31} = -B^j_{55}\alpha$$
$$k^j_{32} = -B^j_{44}\beta$$
$$k^{jk}_{11} = -D^{jk}_{55} - D^{jk}_{11}\alpha^2 - D^{jk}_{66}\beta^2$$
$$k^{jk}_{12} = -(D^{jk}_{12} + D^{jk}_{66})\alpha\beta; \quad k_{21} = k_{12}$$
$$k^{jk}_{22} = -D^{jk}_{44} - D^{jk}_{22}\beta^2 - D^{jk}_{66}\alpha^2$$

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