

# UNIVERSAL CARPET PLOTS FOR STIFFNESS AND STRENGTH OF CARBON/EPOXY LAMINATES

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## ABSTRACT

This paper describes “universal” carpet plots, where a single carpet plot suffices for preliminary design of composite structures using a broad class of materials such as carbon/polymer in general, including all grades of carbon fibers and all varieties of polymer matrices. In addition to carpet plots for stiffness, this paper proposes a novel methodology for developing universal carpet plots for strength and applying them for preliminary design of composite structures. With the availability of universal carpet plots for both stiffness and strength, the designer is free to come up with preliminary designs using a universal material; then change the material selection easily after the preliminary design is completed, just by making minor adjustments. A further advantage is that the proposed carpet plots are based on a small material property dataset.

## 1. INTRODUCTION

A multitude of material systems have been developed to fill the needs of an ever increasing range of applications for composite materials. While diversity of material systems provides unbounded opportunities for matching the perfect material to each application, it also presents a challenge for design, requiring full material characterization of every new material before it has been adopted for a particular application. This type of characterization is onerous and thus unlikely to be comprehensive. Furthermore, most industries rely on fast and flexible preliminary design, to promote early material selection. Detailed analysis is later used to refine the product for prototyping and production. Although computer aided design tools coupled with material databases are gaining popularity, they require a-priori full material characterization. Current software require data for at least five or six different measures of strength (longitudinal tension and compression strengths, transverse tension and compression strengths, as well as in-plane and intralaminar shear strengths). A new material is unlikely to have such comprehensive database until it becomes established, but how is it supposed to become established if the chosen analysis tool does not make it practical to design with it? Furthermore, many designers can complete a preliminary design faster and more cost effectively by using classical design methods coupled with carpet plots of material properties. Until recently, this route required a-priori drafting of carpet plots for every conceivable material system. This presentation will demonstrate the applicability of “universal” carpet plots, where a single carpet plot suffices for preliminary design of composite structures using a broad class of materials such as carbon/polymer in general, including all grades of carbon fibers and all varieties of polymer matrices. In addition to carpet plots for stiffness similar to those proposed in [1,2,3], this presentation proposes a novel methodology for developing universal carpet plots for strength. With the availability of universal carpet plots for both stiffness and strength, the designer is free to come up with preliminary designs using a universal material; then change the material selection easily after the preliminary design is completed, just by making

minor adjustments. These adjustments are based on a small material property dataset that can be generated easily for any new material system. A full material characterization is not necessary until the material is adopted and the detailed analysis stage has been reached.

## 2. LAMINATE MODULI

### 2.1 Orthotropic-plate analogy

For preliminary design, a composite laminate can be analyzed as an orthotropic plate with effective laminate properties  $E_x$ ,  $E_y$ ,  $G_{xy}$ ,  $\nu_{xy}$ . The effective properties can be calculated in terms of the elastic properties of the ply  $E_1$ ,  $E_2$ ,  $G_{12}$ ,  $\nu_{12}$ , plus the laminate stacking sequence (LSS) consisting of a set of ply thicknesses and orientations. The process involves calculation of the extensional-stiffness matrix  $[A]$ , from which the laminate effective properties result [4, (6.42)].

The stiffness matrix of the ply (with units MPa) in the coordinate system 1,2,3 of the ply is calculated in terms of elastic properties of the ply as follows

$$[Q] = \begin{bmatrix} E_1^{-1} & -\nu_{12}E_1^{-1} & 0 \\ -\nu_{12}E_1^{-1} & E_2^{-1} & 0 \\ 0 & 0 & G_{12}^{-1} \end{bmatrix}^{-1} \quad (1.1)$$

The laminate stiffness matrices are denoted by  $[A]$ ,  $[B]$ ,  $[D]$ , and  $[H]$ , but it is convenient to define corresponding thickness-normalized matrices as follows

$$[a] = \frac{1}{t}[A] = \frac{1}{t} \sum_{k=1}^N [\bar{Q}]_k t_k \quad ; \quad [b] = \frac{1}{t^2}[B] \quad ; \quad [d] = \frac{12}{t^3}[D] \quad ; \quad [h] = \frac{1}{t}[H] \quad (1.2)$$

where  $t$  is the thickness of a laminate with  $N$  plies,  $[\bar{Q}]$  is the ply stiffness matrix rotated to the  $x,y,x$  coordinate system of the laminate, and  $A$ ,  $B$ ,  $D$ ,  $H$  are the extensional-, coupling-, bending-, and intralaminar-stiffness matrices, respectively [4, (6.20)].

When the laminate is to be loaded primarily by membrane forces  $N_x$ ,  $N_y$ ,  $N_{xy}$ , the in-plane laminate modulus along the laminate direction  $x$  is calculated as follows

$$E_x = \frac{a_{11}a_{22} - a_{12}^2}{a_{22}} \quad ; \quad E_y = \frac{a_{11}a_{22} - a_{12}^2}{a_{11}} \quad ; \quad G_{xy} = a_{66} \quad ; \quad \nu_{xy} = \frac{a_{12}}{a_{22}} \quad ; \quad G_{xz} = h_{55} \quad (1.3)$$

Note that the most efficient use of a laminate is when it is carrying membrane loads, but if the loading is primarily bending, one should use flexural moduli defined as follows

$$E_x^b = \frac{d_{11}d_{22} - d_{12}^2}{d_{22}} \quad ; \quad E_y^b = \frac{d_{11}d_{22} - d_{12}^2}{d_{11}} \quad ; \quad G_{xy}^b = d_{66} \quad ; \quad \nu_{xy}^b = \frac{d_{12}}{d_{22}} \quad (1.4)$$

If the laminate is balanced/symmetric, then  $[a]$  is identical to the  $[Q]$  matrix of a homogeneous, orthotropic material with elastic properties  $E_x$ ,  $E_y$ ,  $G_{xy}$ ,  $\nu_{xy}$ . Therefore, all the formulas available in

the literature for analysis of orthotropic plates can be used for the design of laminates, simply by using (1.3)--(1.4) in those formulas. For example, the deflection at the tip of a cantilever beam of length  $L$  subject to a concentrated load  $P$  at the tip is

$$\delta = \frac{PL^3}{E_x^b I} + \frac{PL}{\alpha A G_{xz}} \quad (1.5)$$

where  $\alpha$  is the shear correction factor [4, Table 10.1],  $A$ ,  $I$ , are the area and second-moment of the cross-section, respectively, and  $E_x^b, G_{xz}$  are effective moduli defined by (1.3--1.4). If the laminate is not balanced/symmetric, then the use of effective properties is an approximation that can be tolerated during preliminary design. A measure of the deviation from the balanced/symmetric (perfect homogenization) situation is proposed in Section 2.3.

## 2.2 Trace-normalized moduli

The trace of  $[Q]$  is calculated as follows

$$tr(Q) = Q_{11} + Q_{22} + 2Q_{66} \quad (1.6)$$

The trace is invariant with respect to rotations, meaning that

$$tr(\bar{Q}) = tr(Q) \quad (1.7)$$

The trace of  $[Q]$  provides an indication of the bulk-stiffness of the material system. The higher the trace, the higher the volumetric stiffness, which you can relate to the resistance you feel when compressing a tennis ball in your hand. Furthermore, using [4,(6.20)] it is easy to show that  $tr(a)=tr(Q)$  and  $tr(B)=0$ , where  $[B]$  is the bending-extension matrix.

Since  $tr(Q)$  has the same value regardless of orientation, and it is an invariant that represents the incompressibility of the material system (say AS4/3501-6) it is possible to normalize the laminate moduli by the trace, as follows

$$E_x^* = \frac{E_x}{tr(Q)} \quad ; \quad E_y^* = \frac{E_y}{tr(Q)} \quad ; \quad G_{xy}^* = \frac{G_{xy}}{tr(Q)} \quad ; \quad \nu_{xy}^* = \nu_{xy} \quad (1.8)$$

In this way, the bulk-stiffness of the specific material system is removed, leaving intact the dependence of effective properties with respect to the LSS of each particular laminate. Normalization will allow us to calculate normalized properties as a function of LSS for a generic carbon/epoxy system, and then use them for any specific carbon/epoxy system simply by recovering the actual laminate moduli by multiplying by the actual  $tr(Q)$  of the specific material system that we wish to use, as follows

$$E_x = E_x^* \times tr(Q) \quad ; \quad E_y = E_y^* \times tr(Q) \quad ; \quad G_{xy} = G_{xy}^* \times tr(Q) \quad ; \quad \nu_{xy} = \nu_{xy}^* \quad (1.9)$$

For example, using ply properties for AS4/3501-6, we calculate  $tr(Q)$  for AS4/3501-6 and the normalized moduli for the following family of laminates

$$[0_m / \pm 45_p / 90_q]_r \quad (1.10)$$

for various values of  $m$ ,  $p$ ,  $q$ , which are the number of plies in each of those orientations, and  $r$  is the number of repetitions of the sublaminates described between square brackets. The results can be plotted as a function of the %  $\pm 45$  plies in the laminate, with isocurves for various values of % [0] and % [90] plies in the laminate, as shown in Figure 1.

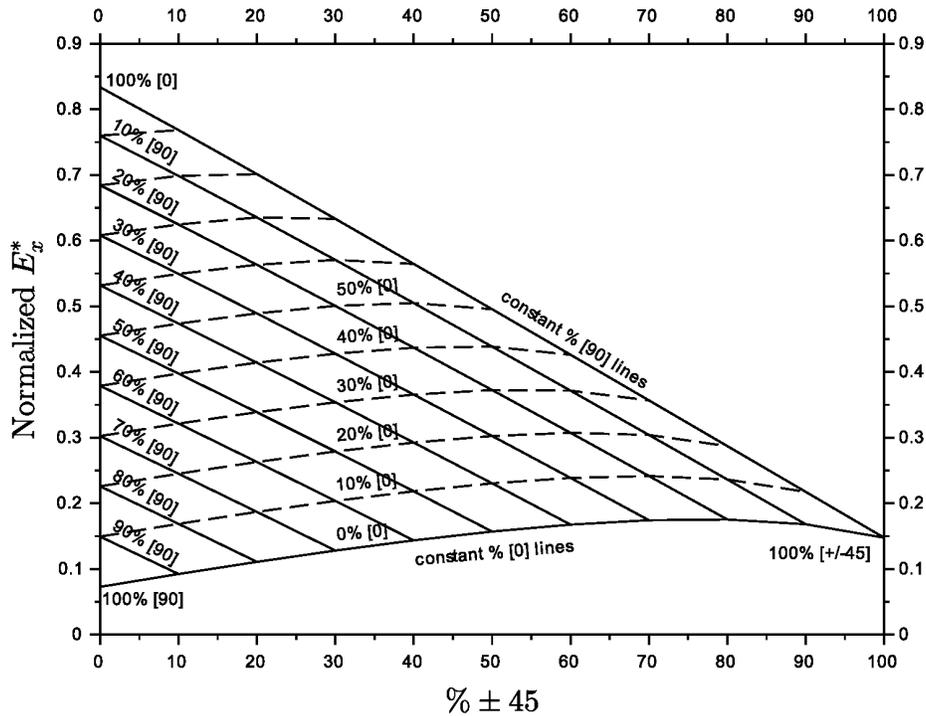


Figure 1. Universal carpet plot for any Carbon/Epoxy laminate. Use (1.9) to estimate actual values for the material system of choice.

We are now ready to recover the actual moduli for any material system, say T300/914C, simply by using (1.9) and the value of  $tr(Q)$  for T300/914C, which can be calculated with (1.6) in terms of the ply properties of T300/914C. Comparison of predicted and actual values for five material systems and both, quasi-isotropic and hard laminates, are shown in Figures 2 and 3.

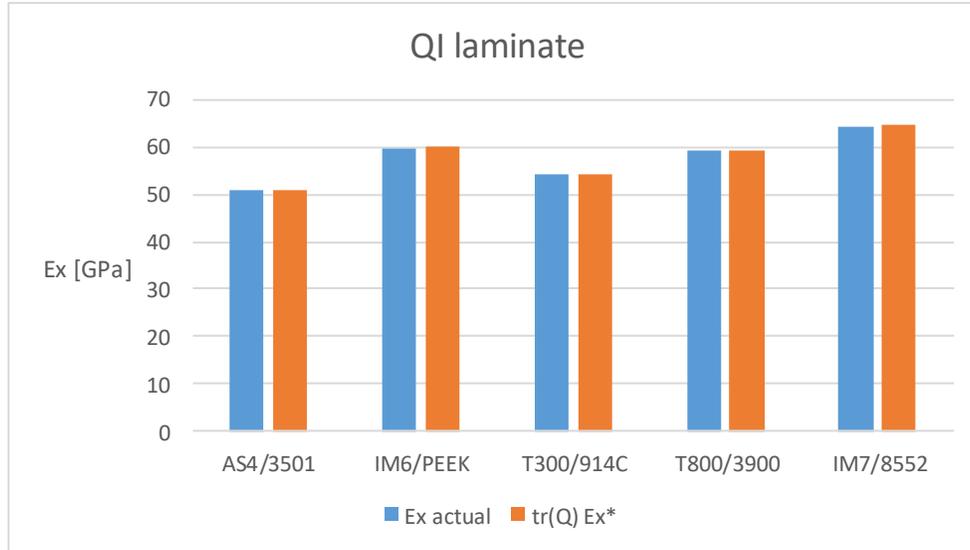


Figure 2. Comparison between actual  $E_x$  and estimated  $E_x=tr(Q)E_x^*$  values for quasi isotropic laminates and five popular material systems.

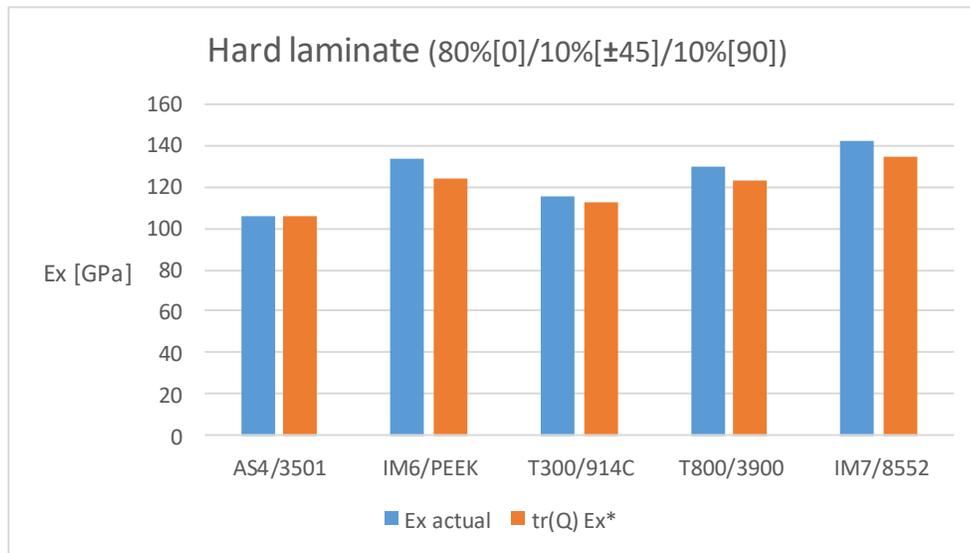


Figure 3. Comparison between actual  $E_x$  and estimated  $E_x=tr(Q)E_x^*$  values for hard laminates (80 % [0] / 10 % [±45] / 10 % [90]) and five popular material systems.

### 2.3 Homogenization

Homogenization is the process of designing a laminate that behaves as much as possible as an orthotropic plate. The homogenization ratio  $h_r$  is defined here to quantify how close a laminate is to perfect homogenization. A balanced/symmetric laminate has  $h_r=1.0$ . For all other laminates, it is defined as follows

$$h_r = 1 - \sqrt{\frac{b_{II}(r)}{b_{II}(r=1)}} \quad (1.11)$$

where  $b_{II}(r)$  is the second invariant of the normalized  $b$ -matrix (1.2) with repetitions  $r$  and  $b_{II}(r=1)$  is the same for repetitions  $r=1$  in (1.10). The second invariant is defined as

$$b_{II} = \frac{1}{2} [tr^2(b) - tr(b^2)] \quad (1.12)$$

Unsymmetric laminates such as (1.10) have coupling effects that are not represented by laminate moduli because the later are exact only for orthotropic (thus symmetric) plates. However, as the number of repetitions  $r$  increases, the coupling effects become negligible. When a unit stress-resultant  $N_x$  is applied to a laminate like (1.10), the membrane strain (solid line) converges rapidly to an asymptotic value that coincides with the membrane strain for the corresponding symmetric laminate. The curvature (dashed line) induced by bending-extension coupling dies out as  $r$  increases. The strain at the surface of the laminate (solid dots), which is the sum of the membrane strain plus the curvature-induced strain, converges to the membrane strain as  $r$  increases. This can be seen better by observing the evolution of the ratio between those strains (open square symbols), which decreases monotonically for increasing  $r$ . That means that as  $r$  increases, the laminate behaves more and more as a symmetric laminate, which would have no curvature under pure membrane load. Furthermore, the open dots curve displaying the homogenization ratio  $h_r$  vs.  $r$  is a mirror image of surface-to-membrane strain (open squares) curve. For  $r=10$ , the homogenization is 90 % complete and the surface-to-membrane strain ratio has concordantly dropped to 1.1.

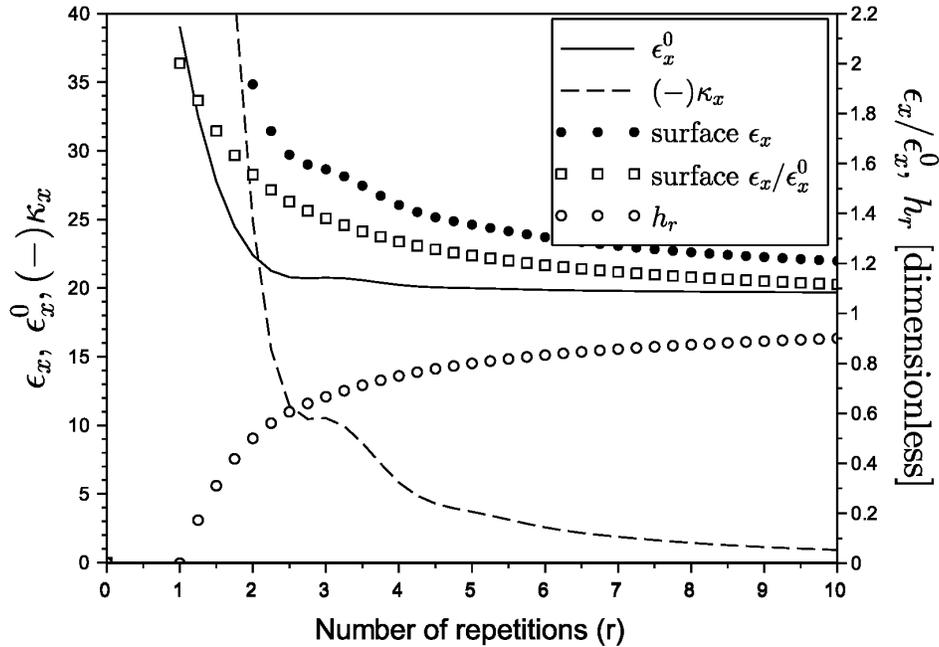


Figure 4. Influence of repetitions  $r$  on coupling-effects of laminates in the family (1.9).

### 3. IN-PLANE STRENGTH

Although First Ply Failure (FPF) and Last Ply Failure (LPF) can be computed with software tools such as [5], to do so requires a priori knowledge of the LSS and material system to be used. That means that calculation of FPF and LPF is an analysis exercise, not a design exercise. For design, one would like to have precomputed laminate strength values for a family of laminates, such as (1.10), so that the engineer can iterate among laminates in such family until a LSS and laminate

thickness can be narrowed down for the particular application at hand. Such precomputed values exist in the form of carpet plots, but traditional carpet plots are specific for each material system. Thus, a change in material system requires a new design from scratch. Ideally, one would like to have carpet plots that are independent of the material system used, so that the LSS can be chosen once and for all, and only the laminate thickness remains to be computed for the final material system chosen. Strength carpet plots that are independent of material system are proposed in Sections 3.1 and 3.2.

### 3.1 Universal Carpet Plots for In-plane Strength

The truncated-maximum-strain (TMS) criterion [6] is proposed here to generate universal carpet plots. Therefore, laminate LPF strength are predicted here using the TMS criterion. An advantage of TMS is that requires only two material properties, the tensile/compressive strain-to-failure of the unidirectional ply ( $\epsilon_{1t}$  and  $\epsilon_{1c}$ ). If these are not available, one can approximate the behavior of the ply as being linear-elastic up to failure, to calculate the required material properties as follows:  $\epsilon_{1t} = F_{1t} / E_1$  and  $\epsilon_{1c} = F_{1c} / E_1$ , where the  $F$ -coefficients are the tensile/compressive strength of the unidirectional ply, which are more easily available. However, strains-to-failure are preferred, if available, in order to more accurately account for any nonlinearity that might be present in the stress-strain curve.

The use of TMS implies that the design must follow the 10% rule or similar, so that at least certain percentage (usually 10%) of the laminate thickness will be represented by all of the following: 0, 90, and  $\pm 45$  plies, or by all orientations in any laminate with complementary laminas, such as  $[\pm 10, \pm 80, \pm 35, \pm 55]_r$ . The use of TMS also implies that complementary laminas (at  $90^\circ$  from each other) exist for all angles. This is necessary to arrest the matrix cracks in one lamina using the stiffness/strength of the fiber in the complementary lamina.

Furthermore, it is assumed that matrix cracking is arrested by the matrix being more compliant than the carbon fibers, or by the fact that knock down factors are applied in the design to take care of matrix cracking, open-hole compression, stress raisers, and eventual repairs. In such scenario, the LPF of the laminate is controlled by the tensile/compressive ultimate-elongation in the fiber direction of the critical lamina. Once a lamina has failed by one of these failure modes (tension/compression/fiber-shear), it is assumed that LPF has been reached or that it is imminent.

In this way, two strength ratios  $R_1$ ,  $R_6$  are calculated as follows

$$\begin{aligned} R_1 &= \langle \epsilon_{1t} / \epsilon_1 \rangle + \langle -\epsilon_{1c} / \epsilon_1 \rangle \\ R_6 &= \frac{(1 + \nu_{12}) \max(\epsilon_{1t}, \epsilon_{1c})}{|\epsilon_1 - \epsilon_2|} \end{aligned} \quad (1.13)$$

where  $\langle x \rangle = (x + \text{abs}(x))/2$  and the overall LPF is given by  $R = \min(R_1, R_6)$ , which flags imminent LPF. The ply strains are calculated with classical lamination theory (CLT).  $R_1$  flags tensile/compressive failure in the fiber directions, and  $R_6$  flags fiber shear failure. The failure envelope resembles, but it is not quite the same, as the Tresca criterion. In-plane shear load induces tension/compression in the off-axis plies. If, as it is usually the case, the ply compressive strength

is smaller than its tensile strength, shear failure is controlled by compression in some of the off-axis plies. Taking all of this into account, the laminate strength can be normalized as follows

$$F_{xt}^* = \frac{F_{xt}}{E_1 \epsilon_{1t}} \quad ; \quad F_{xc}^* = \frac{F_{xc}}{E_1 \epsilon_{1c}} \quad ; \quad F_{xy}^* = \frac{F_{xy}}{E_1 \epsilon_{1c}} \quad (1.14)$$

Note that only tensile failure of the laminate is controlled by ply tensile strength, while both compression and shear strengths of the laminate are controlled by compression strength of the ply, making the later property the most critical for design [7].

Applying a uniaxial membrane load  $N=[\pm I, 0, 0]$ , the tensile/compressive strength of a laminate is detected with an appropriate failure criterion (1.13) when the load is increased by a factor  $R$ , called strength ratio or by a factor  $1/I$ , where  $I$  is called failure index. Most failure criteria can be set up to calculate either  $R$  or  $I$  or both. The laminate tensile/compressive strength are then calculated as  $F_{xt}=R/t$  (tensile) and  $F_{xc}=R/t$  (compressive), respectively, where  $t$  is the thickness of the laminate. To calculate  $F_{yt}$  and  $F_{yc}$ , one can simply rotate the laminate  $90^\circ$  and use the procedure just described. In fact, only the carpet plot for  $F_{xt}$  and  $F_{xc}$  is needed. Values of  $F_{yt}$ ,  $F_{yc}$  can be read from the same plot by swapping the % [0] and % [90] values while reading from the plot.

Applying an in-plane shear load  $N=[0, 0, \pm I]$ , the in-plane shear strength of a laminate is detected likewise with an appropriate failure criterion (1.13) when the load is increased by a factor  $R$ . The laminate in-plane shear strength is then calculated as  $F_{xy}=R/t$ , where  $R$  is  $R=\min(R_I, R_\delta)$ . Note that  $R_\delta$  in (1.13) is not an indication of matrix shear failure, but an indication of fiber shear failure in the critical ply, for any laminate loading type: tensile, compressive, or shear.

Both tensile/compressive modes can be plotted in a single carpet plot, as in Figure 5, but it must be noted that the calculations of actual tensile/compressive strength values require the use of different normalization factors, i.e.,

$$F_{xt} = F_{xt}^* E_1 \epsilon_{1t} \quad ; \quad F_{xc} = F_{xc}^* E_1 \epsilon_{1c} \quad ; \quad F_{xy} = F_{xy}^* E_1 \epsilon_{1c} \quad (1.15)$$

The in-plane shear strength of the laminate is only a function of the amount of  $\pm 45$  present in the laminate, so it can be expressed by a formula rather than a carpet plot

$$F_{xy}^* = 0.3524 \frac{x}{100} + 0.0884 \quad (1.16)$$

where  $x$  is the percentage  $\pm 45$  in the LSS. This does not mean that the [0] and [90] plies do not contribute to the load carrying capacity, but only means that the carpet plot can be collapsed into a line, which is then represented by a linear equation such as (1.16).

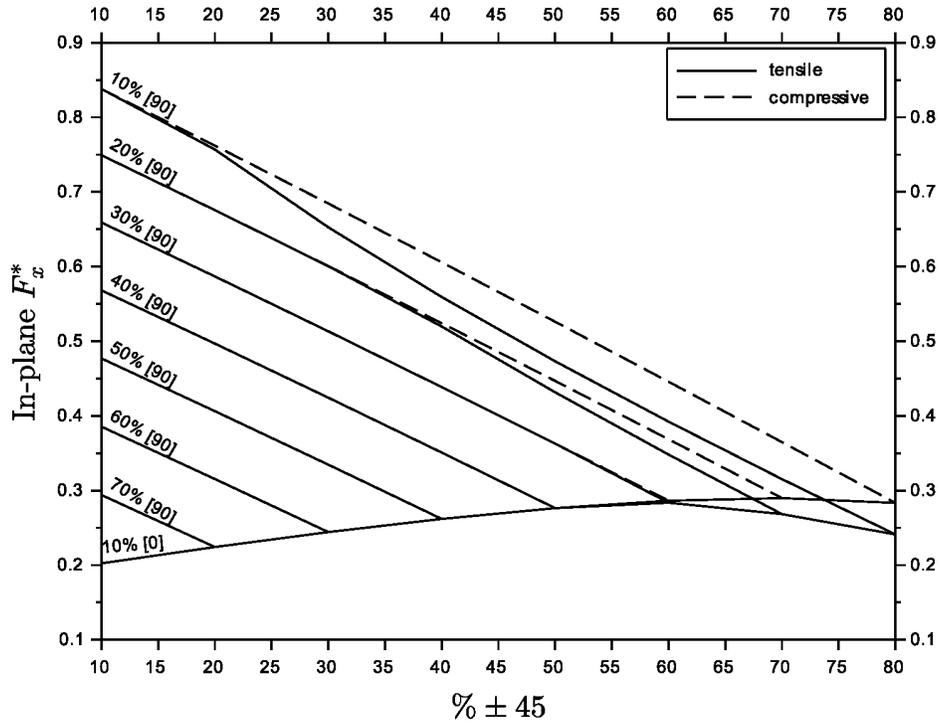


Figure 5. Universal carpet plot for in-plane tensile/compressive strength of carbon/epoxy laminates.

### 3.2 Universal Carpet Plots for Flexural Strength

For illustration, consider a symmetric laminate under bending stress-resultant  $M=[\pm 1, 0, 0]$ . When the applied moment is positive, the plies located farther away from the midsurface are in compression if above the midsurface and in tension if below it. If the ply compression strain-to-failure is smaller than the ply tensile strain-to-failure, as it usually is the case for modern material systems, failure is controlled by compression. Now reverse the sign of the applied moment, and you see that failure is still controlled by compression but the critical ply moved from above to below the midsurface. That means that flexural strength is the same regardless of the sign of the applied moment and furthermore it is always controlled by compression.

Next apply a twisting moment  $M=[0, 0, \pm 1]$ . Again, some off-angle ply above/below the midsurface will be in tension/compression and the ply under compression controls the failure. Now, if the sign of the applied twisting load is reversed, the critical ply moves from top to bottom or vice versa but the failure occurs at the same magnitude of applied load. That confirms the fact that even under twisting load, the flexural strength is the same regardless of the sign of the applied moment and furthermore it is always controlled by compression.

For non-symmetric laminates, as (1.10), the previous discussion applies when the number of repetitions  $r$  increases. As the homogenization increases, the ratio of individual ply thickness to overall laminate thickness  $t$  decreases and the critical plies above and below the midsurface tend to be at almost the same distance from the midsurface. Under those conditions, the flexural strength

is almost the same regardless of the sign of the applied moment and it remains controlled by compression.

Applying a bending moment (per unit width of plate)  $M=[\pm 1, 0, 0]$ , the flexural strength of a laminate is detected with an appropriate failure criterion (1.13) when the load is increased by a factor  $R$ . Since the bending stress at the outer surface of the laminate can be calculated as  $\sigma_x = 6M_x / t^2$ , the laminate flexural strength is then calculated as  $F_x^b = 6R/t^2$ , where  $R = \min(R_1, R_6)$  and  $t$  is the laminate thickness. Likewise, applying a twisting moment (per unit width of plate)  $M=[0, 0, \pm 1]$ , results in LPF for certain value or  $R$ , which leads us to the laminate twisting strength  $F_{xy}^b = 6R/t^2$ . When these calculations are performed for a variety of LSS, the results can be plotted in a universal carpet plot as shown in Figure 6.

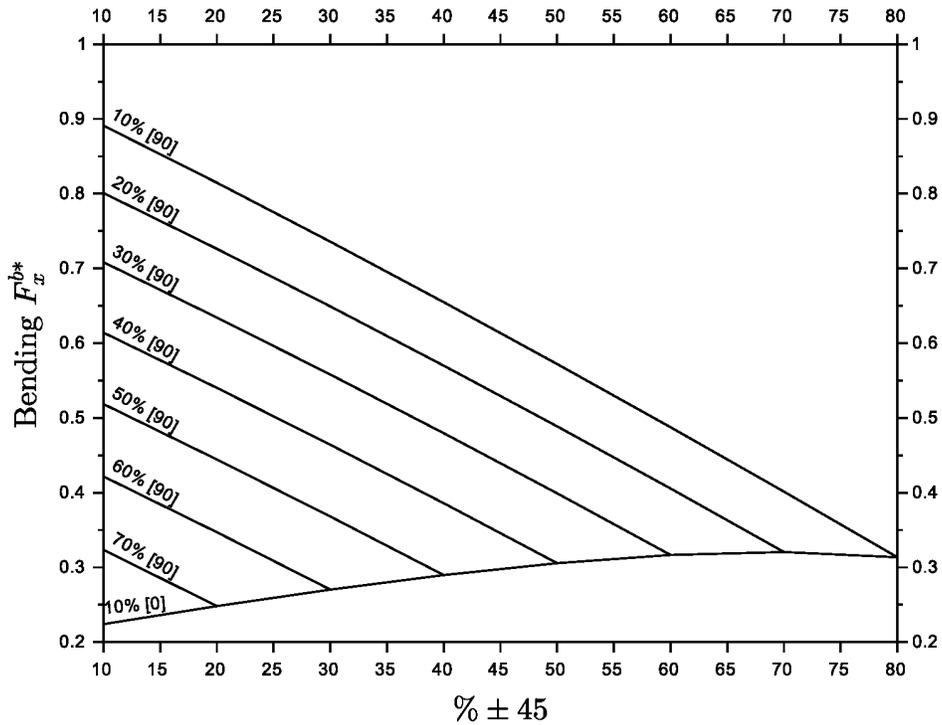


Figure 6. Universal carpet plot for flexural strength of carbon/epoxy laminates.

Note the striking resemblance between Figures 5 (compression) and Figure 6. This is because in both cases the failure is controlled by ply-compression and independent of the sign of the applied load. This does not mean that a laminate in bending can carry load as efficiently as one loaded by membrane loads, because the flexural strength is only reached on the surface of a laminate in bending while for a laminate loaded in-plane, the tensile/compressive/shear strength is carried throughout the entire thickness of the laminate, thus making it a much more efficient use of the material.

The twisting flexural strength of the laminate is only a function of the amount of  $\pm 45$  present in the laminate, so it can be expressed by a formula rather than a carpet plot

$$F_{xy}^{b*} = 0.3661 \frac{x}{100} + 0.0866 \quad (1.17)$$

where  $x$  is the percentage  $\pm 45$  in the LSS. This does not mean that the [0] and [90] plies do not contribute to the load carrying capacity, but only means that the carpet plot can be collapsed into a line, which is then represented by a linear equation. Again, note the striking similarity between (1.16) and (1.17), due to the reasons given above.

#### 4. CONCLUSIONS

Universal carpet plots provide a simple tool to estimate elastic moduli and strength of laminates as a function of percentage of  $\pm 45$ , 0, and 90 plies without getting too involved with the calculations. All that is needed to transform normalized values from the plots into actual values are  $tr(Q)$  and the two values of tensile/compressive longitudinal strain-to-failure of the actual material system. The former can be calculated easily or found in material property tables such those available in [4, Ch. 1]. In this way, the engineer can concentrate on the preliminary design and only later seek a refined analysis using more sophisticated tools once the material system, LSS, and laminate thickness have been tentatively selected. The methodology can be applied to other material systems but careful consideration must be given to the assumptions made for the application of the failure criteria chosen.

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