ASYMPTOTIC POST-BUCKLING FEM ANALYSIS OF LAMINATED COMPOSITE FOLDED PLATES

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Abstract
Koiter's asymptotic approach is a powerful alternative to path-following techniques in the analysis of slender structures. Its reliability for the evaluation of post-critical behavior, including strong nonlinear pre-critical and modal interaction, is well know. The most recent finite element implementations of Koiter's approach are based on corotational formulation, which allows to automatically reuse linear finite elements into a nonlinear context once the geometrically nonlinear corotational kinematics are developed. Beam assemblages, folded plates structures are extended here to laminated composite slender structures. By reusing the mixed finite element MISS-4 into a corotational framework, an analysis of folded plate structures within an FSDT theory is developed. The analysis of a box structure is presented to highlight the accuracy of the Koiter analysis in the recovery of buckling and post-buckling behavior with low computational cost.

1. INTRODUCTION

Koiter's asymptotic analysis is a powerful tool for the analysis of geometrically nonlinear structures. It is an effective and accurate strategy for predicting the initial post-critical behavior in both cases of limit or bifurcation points, as well as mode interaction [1]. Koiter's approach is based on a fourth-order expansion of the strain energy. The book [2] contains past and recent research efforts. A careful tuning of both the continuum model and its finite element implementation as well as a coherent evaluation of the kinematic relationships are needed to obtain accurate results. Moreover, a structural model that is geometrically exact to fourth-order is needed. This is a very strict requirement, but it is important for the reliability which are very sensitive to the correctness of the energy expression.

The corotational approach (CR) proposed by Garcea et al. [3] is a tool to obtain an objective model starting from a linear model. In the same paper the general approach has been applied successfully to the analysis of 3D beam assemblages.

Recently, Zagari et al. [4] have been proposed an application to folded plates and shell structures. The linear finite element MISS-4 [5] is used in a corotational framework. MISS-4 is a simple four node quadrilateral high performance finite element based on Hellinger-Reissner variational formulation. The displacement interpolation is ruled by 24 DOFs (3 displacements and 3 rotations for nodes) while the stress is self equilibrated and isostatic using 18-β stress parameters. The drilling rotations are introduced á la Allman, and spurious energy modes are avoided.

In the present paper the recent developments of Koiter analysis for the analysis of laminated composite plates [6] are discussed. The employed plate model is the first-order shear deformation theory (FSDT) [7]. Using MISS-4 finite element within the corotational formulation the first four order of variation of the strain energy, needed by Koiter analysis, have been evaluated. The robustness and accuracy have been tested in the buckling and postbuckling analyses of clamped box under shear forces.

2. LINEAR FLAT SHELL ELEMENT

We assume that the initial reference configuration of the element is flat and referred to a local Cartesian frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. We denote by $\{x, y\}$ the position of the vector along the middle surface Ω lying in the plane defined by the unit vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$, by s the thickness along the $\boldsymbol{e}_3$ direction and by Γ the boundary of Ω. Using FSDT [7], the mixed Hellinger-Reissner strain energy of the flat shell can be written as: The mixed strain energy for a flat shell can be written as:

$$\Phi[\boldsymbol{t}, \boldsymbol{d}] = \int_\Omega \left\{ \boldsymbol{t}^T \boldsymbol{D} \boldsymbol{d} - \frac{1}{2} \boldsymbol{t}^T \boldsymbol{E}^{-1} \boldsymbol{t} \right\} \mathrm{d}\Omega, \quad \boldsymbol{t} = \begin{bmatrix} \boldsymbol{t}_m \\ \boldsymbol{t}_f \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} \boldsymbol{d}_m \\ \boldsymbol{d}_f \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_m & \cdot \\ \cdot & \boldsymbol{D}_f \end{bmatrix} \tag{1}$$

where vectors $\boldsymbol{t}_m$ and $\boldsymbol{t}_f$ collect, respectively, the membrane stress resultants and the moment and shear resultants, and $\boldsymbol{d}_m$ and $\boldsymbol{d}_f$ collect, respectively, the in- out-plane kinematical parameters

$$\boldsymbol{t}_m = \begin{bmatrix} N_x \\ N_y \\ N_{xy} \end{bmatrix} \quad , \quad \boldsymbol{t}_f = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \\ S_x \\ S_y \end{bmatrix} \quad , \quad \boldsymbol{d}_m = \begin{bmatrix} d_x \\ d_y \end{bmatrix} \quad , \quad \boldsymbol{d}_f = \begin{bmatrix} d_z \\ \varphi_x \\ \varphi_y \end{bmatrix} \tag{2}$$

The differential operators $\boldsymbol{D}_m$ and $\boldsymbol{D}_b$ are defined as

$$\boldsymbol{D}_m = \begin{bmatrix} \partial/\partial x & \cdot \\ \cdot & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \quad , \quad \boldsymbol{D}_f = \begin{bmatrix} \cdot & \cdot & -\partial/\partial x \\ \cdot & \partial/\partial y & \cdot \\ \cdot & \partial/\partial x & -\partial/\partial y \\ \partial/\partial x & \cdot & 1 \\ \partial/\partial y & -1 & \cdot \end{bmatrix} \tag{3}$$

The matrix of elastic coefficients, $\boldsymbol{E}$ can be written as

$$\boldsymbol{E} = \begin{bmatrix} \boldsymbol{E}_m & \boldsymbol{E}_{mf} \\ \boldsymbol{E}_{mf}^T & \boldsymbol{E}_f \end{bmatrix}. \tag{4}$$

Denoting by $\boldsymbol{E}_m$ and $\boldsymbol{E}_f$ the membrane and the flexural behavior, respectively, and by $\boldsymbol{E}_{mf}$ the membrane/flexural coupling depending by lamina material properties and laminate stacking sequence (LSS) (see [7] for details). When the stress resultants are defined so that the bulk equilibrium equations are satisfied with zero load, the following identity holds:

$$\int_\Omega \boldsymbol{t}^T \boldsymbol{D}\boldsymbol{d}\, \mathrm{d}\Omega = \int_\Gamma \boldsymbol{t}^T \boldsymbol{N}^T \boldsymbol{d}\, \mathrm{d}\Gamma = \int_\Gamma \boldsymbol{t}_m^T \boldsymbol{N}_m^T \boldsymbol{d}_m\, \mathrm{d}\Gamma + \int_\Gamma \boldsymbol{t}_f^T \boldsymbol{N}_f^T \boldsymbol{d}_f\, \mathrm{d}\Gamma \tag{5}$$

being $\boldsymbol{N}$ the matrix collecting the components of the unit outward normal to the contour Γ, that can be split into membrane $\boldsymbol{N}_m$ and bending $\boldsymbol{N}_f$ parts.

2.1. Stress and displacements interpolations

Assuming a mixed interpolation for the stress resultants and displacements

$$\boldsymbol{t} = \boldsymbol{B}\boldsymbol{t}_e \quad , \quad \boldsymbol{d} = \boldsymbol{U}\boldsymbol{d}_e \tag{6}$$

where $\boldsymbol{B}$ is the matrix collecting the assumed stress modes, $\boldsymbol{t}_e$ is the vector of stress parameters, $\boldsymbol{U}$ is the matrix of the displacement shape functions and $\boldsymbol{d}_e$ is the vector of the displacement and rotation kinematical parameters. Substituting (6) into (1) and integrating on the element domain Ω_e lead to the evaluation of the element mixed energy

$$\Phi_e[\boldsymbol{t}_e, \boldsymbol{d}_e] = \boldsymbol{t}_e^T \boldsymbol{D}_e \boldsymbol{d}_e - \frac{1}{2}\boldsymbol{t}_e^T \boldsymbol{H}_e \boldsymbol{t}_e \quad , \quad \begin{cases} \boldsymbol{D}_e = \int_{\Omega_e} \left\{ \boldsymbol{B}^T \boldsymbol{D}\boldsymbol{U} \right\} \mathrm{d}\Omega \\ \boldsymbol{H}_e = \int_{\Omega_e} \left\{ \boldsymbol{B}^T \boldsymbol{E}^{-1} \boldsymbol{B} \right\} \mathrm{d}\Omega \end{cases} \tag{7}$$

with $\boldsymbol{D}_e$ and $\boldsymbol{H}_e$ being the compatibility and flexibility operators, respectively. Their evaluation for MISS-4 finite element is reported in [5].

3. GEOMETRICALLY NONLINEAR FORMULATION

A linear shell finite element can be made geometrically nonlinear through the use of an appropriate corotational algebra that governs the rigid body motion, like the one introduced in [3]. An appropriate corotational frame is defined for each element through the element rotation vector $\boldsymbol{\alpha}_e$ which is a function of the element kinematical parameters $\boldsymbol{d}_e$ in the fixed frame:

$$\boldsymbol{\alpha}_e = \boldsymbol{\alpha}_e[\boldsymbol{d}_e] \tag{8}$$

The local kinematical parameters $\bar{\boldsymbol{d}}_e$ in the CR frame can be related to $\boldsymbol{d}_e$ by a geometrical transformation:

$$\bar{\boldsymbol{d}}_e = \boldsymbol{g}[\boldsymbol{d}_e] \tag{9}$$

Based on the above relations, the linear finite element characterized by energy (7) can be transformed into a geometrically nonlinear element simply by introducing a corotational description and assuming that the element kinematical parameters in eq. (7) are referred to the corotational frame. This leads to:

$$\Phi_e[\boldsymbol{t}_e, \boldsymbol{d}_e] = \boldsymbol{t}_e^T \boldsymbol{D}_e \boldsymbol{g}[\boldsymbol{d}_e] - \frac{1}{2} \boldsymbol{t}_e^T \boldsymbol{H}_e \boldsymbol{t}_e \tag{10}$$

To apply the asymptotic approach to the corotational version of element MISS-4, explicit expressions for the second-, third- and fourth–order energy variations with respect to a configuration which can be either the initial or the bifurcation one, have to be computed. Making use of an appropriate configuration updating process, we can always refer to a configuration characterized by $\boldsymbol{d}_e = \boldsymbol{0}$.

The corotational approach is very convenient to express the strain energy variations, because the only nonlinearity is limited to the geometrical relationship $\boldsymbol{g}[\boldsymbol{d}_e]$, eq. (9). The Taylor expansion of this relationship can be written as

$$\boldsymbol{g}[\boldsymbol{d}_e] = \boldsymbol{g}_1[\boldsymbol{d}_e] + \frac{1}{2}\boldsymbol{g}_2[\boldsymbol{d}_e, \boldsymbol{d}_e] + \frac{1}{6}\boldsymbol{g}_3[\boldsymbol{d}_e, \boldsymbol{d}_e, \boldsymbol{d}_e] + \frac{1}{24}\boldsymbol{g}_4[\boldsymbol{d}_e, \boldsymbol{d}_e, \boldsymbol{d}_e, \boldsymbol{d}_e] + \cdots \tag{11}$$

where $\boldsymbol{g}_n$ are n–multilinear symmetric forms which express the nth Fréchet variations of function $\boldsymbol{g}[\boldsymbol{d}_e]$. Starting from Eq. 11 the high order energy variations needed for Koiter analysis can be easily evaluated.

4. NUMERICAL RESULTS: CLAMPED BOX UNDER SHEAR FORCES

The buckling and post-buckling analysis of a clamped box under shear forces are presented and a comparison with Riks path-following analysis using ABAQUS [8, 9] is made. The geometry, boundary conditions, and load are shown in Fig. (1). The geometrical data are l = 1000 mm, a = 100 mm, r = 250 mm and the thickness is t = 10 mm. The line load is q = 25 kN/mm. The following laminate stacking sequence is considered: $[15/-30/15/-30]_s$. The elastic modula for each lamina are: E_1 = 104.0 GPa, E_2 = 10.3 GPa, G_{12} = 5.15 GPa and ν_{12} = 0.021.

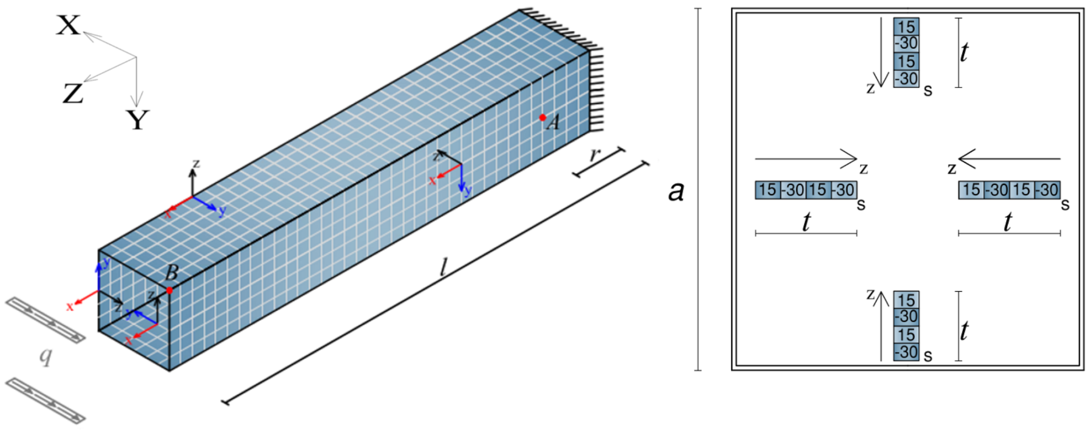

Figure 1. Geometry, boundary conditions, and load for clamped box subjected to shear load.

The critical loads are listed in Table 1 and the buckling modes are shown in Fig. 3. Note that h^2 convergence is achieved for critical values as shown in Fig. 2.

The equilibrium paths are reported in Figs. 5 and 4, while a deformed configuration is reported in Fig. 4. A comparison with Riks analysis is graphed. The accuracy of Koiter approach is clear, both in pre-critical and post-critical behavior, but with lower computational cost.

mesh	λ_1	λ_2	λ_3	λ_4
2	146.72	152.72	202.10	220.47
4	82.87	101.43	131.52	133.71
8	56.56	59.39	77.11	80.06
16	43.49	45.31	54.81	56.96
20	41.89	43.54	52.15	54.02
24	41.01	42.57	50.69	52.41
24 (S8R)	40.61	42.31	50.24	52.07

Table 1. Clamped box under shear forces. Convergence of buckling loads with mesh refinement. The values on the first column refer to the numbers of the element along a. The first column indicates the number of elements used for the mesh along the side. The mesh refining has been done by splitting.

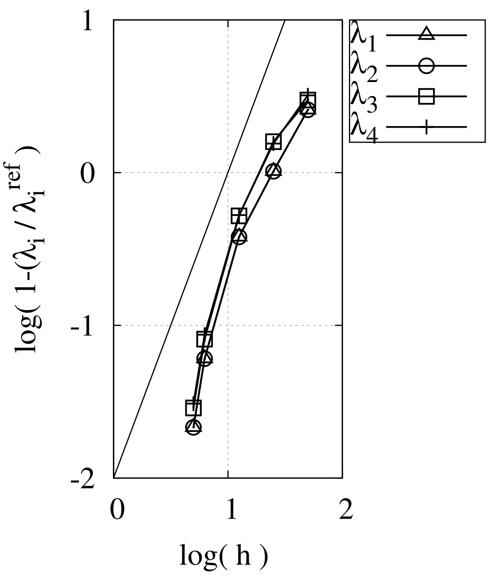

Figure 2. Clamped box under shear forces. Convergence of buckling loads with mesh refinement. The solid line represents h^2 (for reference).

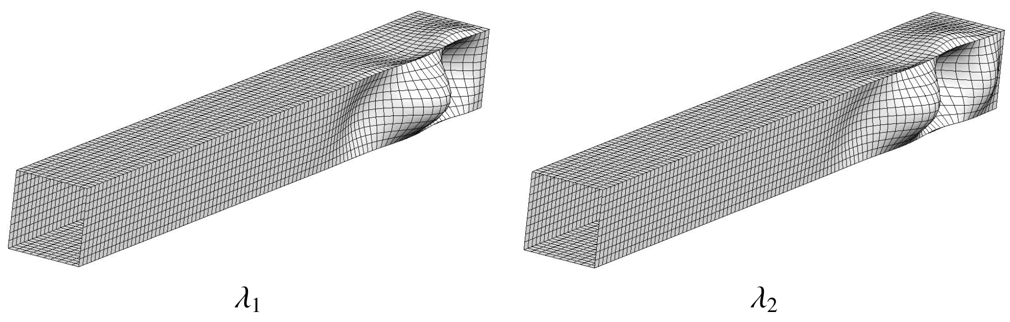

Figure 3. Clamped box under shear forces. Buckling modes corresponding to buckling loads λ_1, λ_2.

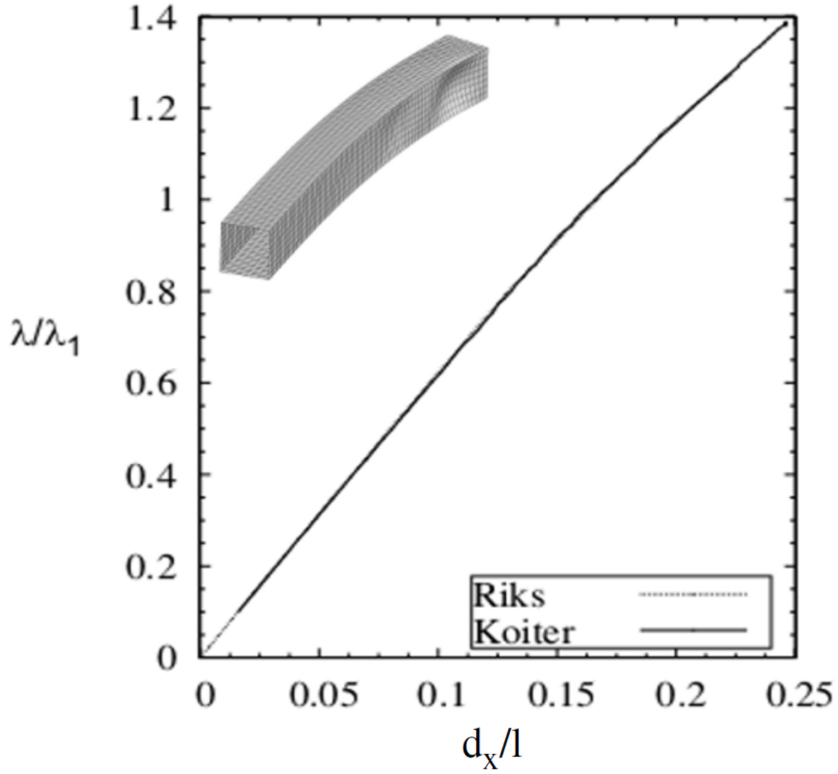

Figure 4. Clamped box under shear forces. Deformed configuration along the equilibrium path at $\lambda = 52.20$ and equilibrium paths evaluated for both Koiter and Riks analyses. The graphed displacement d_x refers to the transversal displacement of point *B* (see Fig. 1).

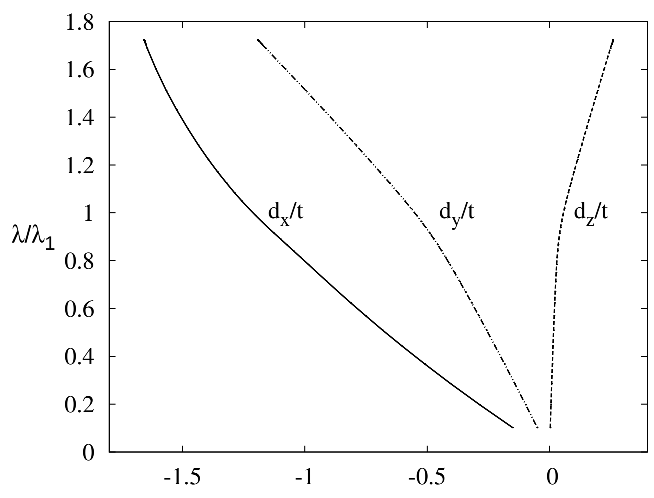

Figure 5. Clamped box under shear forces. Equilibrium path recovered using Koiter analysis. The graphed displacements d_x, d_y and d_z refer to the displacements of point *A* (see Fig. 1).

5. CONCLUDING REMARKS

Koiter's asymptotic analysis represents a valid, less computational expensive alternative to Riks path-following analysis for the recovery the initial post-critical behavior of composite structures, even those displaying strong precritical behavior and buckling mode interaction. Its use in the context of laminated composite folded plate (shell) structures has been discussed. The accuracy of the proposed element has been checked and the convergence of the critical and post-critical quantities show good performance.

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