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A Mechanistic Model for Transverse Damage Initiation, Evolution, and Stiffness Reduction in Laminated Composites

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ABSTRACT

A constitutive model to predict stiffness reduction due to transverse matrix cracking is derived for laminae with arbitrary orientation, subject to in-plane stress, embedded in laminates with symmetric but otherwise arbitrary laminate stacking sequence. The moduli of the damaged laminate are a function of the crack densities in the damaging laminae, which are analyzed one by one. The evolution of crack density in each lamina is derived in terms of the calculated strain energy release rate and predicted as function of the applied load using a fracture mechanics approach. Unlike plasticity-inspired formulations, the proposed model does not postulate damage evolution functions and thus there is no need for additional experimental data to adjust material parameters. All that it is needed are the elastic moduli and critical energy release rates for the laminae. The reduction of lamina stiffness is an integral part of the model, allowing for stress redistribution among laminae. Comparisons with experimental data and some results from the literature are presented.

KEYWORDS

Transverse cracking, Analytical modeling, Computational modeling, Damage mechanics

1. INTRODUCTION

If a laminate is well designed, fiber failure modes, either tensile or compressive, should control the ultimate load of the laminate [1]. However, matrix cracks are often the first degradation event and they trigger other damage modes such as delamination, fiber-matrix debonding, and

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fiber breakage that lead to fracture. Furthermore, matrix cracks drastically increase the permeability of the material, thus allowing access to liquid and gas contaminants that may degrade the fibers and the fiber-matrix interface. Also, matrix cracks facilitate leakage of liquid and gas that would otherwise be contained in tanks, pressure vessels, and other similar structures. In addition, matrix cracks lead to stiffness reduction and stress redistribution to adjacent laminae, which are needed for computation of fiber-dominated failure modes and laminate strength. Therefore, prediction of matrix cracking initiation and evolution remains relevant for the analysis of laminated composites.

Popular methods such as ply discount ignore the gradual redistribution of stress among laminae, which may lead to numerical instability of the solution algorithm. While degradation factors [1] smooth out the sudden stiffness changes brought about by ply discount, they introduce an undesirable empirical parameter, the degradation factor, which changes with laminate stacking sequence (LSS) and material properties, and introduce an artificial residual stiffness for the cracking laminae, equal in magnitude to the degradation factor times the undamaged stiffness of the lamina.

Usually, unidirectional loading of cross-ply laminates produces matrix cracking of the 90° lamina when the load is applied in 0°-direction [5,6,27,41,44,45], but this not the only case. Varna et al. [2] found in experiments for balanced $[0/\pm\theta_4/0_{1/2}]_s$ laminates that matrix cracks appear for angles as low as θ=40°. Yokozeki et al. [3,4] observed matrix cracking for ply angles as low as 30° for unbalanced $[0/\theta/90]_s$ laminates. In addition, matrix cracking has been shown to promote further matrix cracks in adjacent plies [5,6].

Popular stress analysis methods for laminated composites are based on strength criteria, which need experimental lamina strength values, such as transverse strength F_{2T} and shear strength F_6. But these are not true material properties because their values change as a function of the thickness of the lamina and the laminate stacking sequence (both orientation and thickness) of the remaining laminae in a laminate. In-situ values have been proposed to ameliorate this deficiency [7,8], but even if in-situ values bring strength criteria closer to experimentally observed stress/strain at crack initiation, strength criteria do not provide information about crack evolution. Indeed, they need to be supplemented by empirical damage evolution (kinetic) equations written in terms of additional material parameters to be determined by additional experimentation, and thus additional cost [9—18]. Energy criteria are preferable to strength criteria because they are based on invariant material properties [19,20].

Continuum Damage Mechanics (CDM) can be used for prediction of stiffness reduction in the presence of several damage modes [9—18]. In CDM, damage is represented by internal state variables, and evolution equations for these variables are postulated in terms of additional material parameters. A disadvantage of CDM is the difficulty in obtaining these material parameters from experimental data. This issue have been partially solved by combining CDM with other techniques such as micromechanics [21-24], fracture mechanics [25], and in-situ damage effective functions [26].

Another approach is the analytical solution of the stress and strains in a representative volume cell [27,28], an approach that is particularly interesting because the material properties of the damaged laminate depend exclusively on the crack density and no additional parameters or functions are needed. Furthermore, stress transfer methods have been proposed [29—37] but the solutions given are limited to particular cases. In [27] the Laminate Stacking Sequence (LSS) is restricted to $[0/90_n]$s and in [47] to $[\pm\theta/90]_S$, but in both cases only the 90-lamina can damage. In [28] the LSS is restricted to $[\theta_n/\varphi_m]$s with cracks on both directions. In [35] the LSS is restricted to $[0/45]_S$. Although symmetric laminates were analyzed in [19,20], the methodology proposed herein proves to be simpler and less computationally demanding. In [38] a solution is presented for two sets of arbitrarily oriented cracking laminae with angles θ_1 and θ_2 and crack densities λ_1 and λ_2, embedded in a laminate with arbitrary LSS, by using oblique coordinates as in [28]. However, the solution for two damaging laminae is very difficult to generalize to multiple damaging laminae. Furthermore, oblique coordinates introduce numerical instabilities for certain material systems when the angle between the two damaging laminae approaches zero or π. These problems are eliminated with the formulation proposed herein.

Composites Part B, 41:124—132, 2010.

Several criteria have been proposed to predict matrix cracking evolution as a function of the applied load [10,19,20,39–42]. In this study, damage due to matrix cracks of a laminate with symmetric, but otherwise general LSS is predicted by a combination of an analytical solution for the damage activation function *g* and a return mapping algorithm (RMA) to restore equilibrium upon damage. The shear lag analysis is inspired by the work in [28] but developed for a single lamina using a coordinate system aligned with the crack direction and computing damage evolution to one lamina at a time. Furthermore, the proposed formulation takes into account the stiffness of the remaining laminae by a laminate homogenization method similar to that of [19,20]. However, the proposed formulation differs in a number of ways including the method of computation of damage variables $D_2(\lambda_k)$ and $D_6(\lambda_k)$ from the crack densities λ_k in each lamina. Finally, a modified RMA allows the algorithm to reconcile the damage in all laminae yielding corrected values for the damage activation function $g(\lambda_k)$ in each lamina. Using concepts of CDM, the failure criteria of [40] is recast here as a damage activation function. By using the calculated laminate stiffness reduction, the proposed damage activation function automatically and accurately represents the hardening behavior of the cracking lamina embedded in a laminate with correct representation of the constraining effect of adjacent laminae. Unlike plasticity-inspired damage evolution models, the present formulation does not use any additional parameters; all that is needed are the moduli and critical ERR values of the laminae.

2. ANALYTICAL SOLUTION FOR A CRACKING LAMINA

Consider a symmetric laminate with otherwise arbitrary LSS. We are interested in detecting crack initiation, predicting crack evolution and stiffness reduction. A damage activation function can accomplish detection of crack initiation and prediction of crack evolution simultaneously if properly set up in terms of the reducing stiffness of the laminate. Therefore, the proposed analytical solution has two distinct parts: calculation of the damage variables $D_2(\lambda)$ *and* $D_6(\lambda)$ that represent transverse and shear stiffness reduction as a function of crack density λ, and calculation of the damage activation function $g(\lambda) \leq 0$ delimiting the damage and no-damage domains.

Composites Part B, 41:124—132, 2010.

2.1. Stiffness Reduction

An analytical solution is derived in this section to calculate the reduction in transverse and shear stiffness of the laminate as a function of the crack density λ in one lamina. A repetitive unit cell (RUC) is chosen as the volume enclosed by the mid-surface and top-surface of the laminate, the surfaces of two consecutive cracks, and a unit length along the parallel cracks (Figure 1). The length of the RUC is 2*l* and it is related to crack density as

$$2l = 1/\lambda \tag{1}$$

A material coordinate system denoted by x_1-x_2-x_3 is used, where the x_1 and x_2 are in-plane coordinates aligned and perpendicular to the fibers of lamina *k*, respectively.

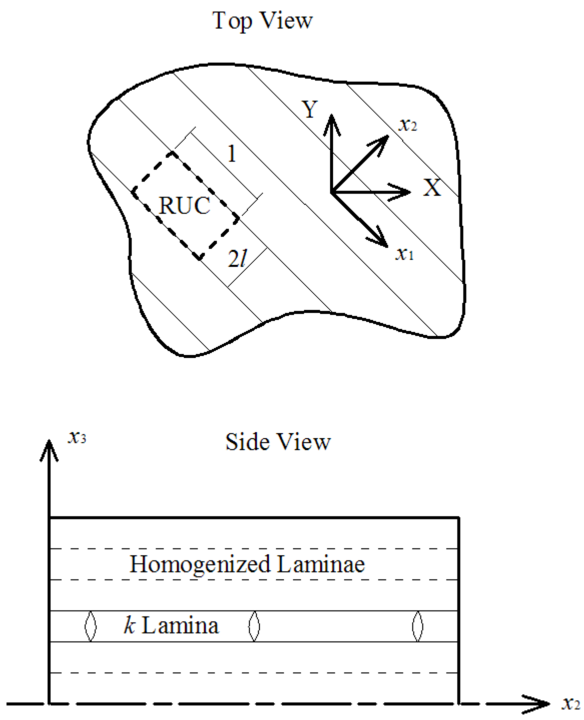

Figure 1. Representative unit cell (RUC) and coordinate systems used.

Composites Part B, 41:124—132, 2010.

Due to the stress-free conditions on the top and bottom surfaces of a thin laminate we have

$$\sigma_3^{(i)} = 0 \tag{2}$$

Due to symmetry of the LSS and load we have

$$\frac{\partial w^{(i)}}{\partial x_1} = \frac{\partial w^{(i)}}{\partial x_2} = 0 \tag{3}$$

where $w^{(i)}$ is displacement in the x_3 direction of the i^{th} lamina. Finally, a linear variation of shear stress in the x_3 direction is assumed in each lamina

$$\begin{aligned} \tau_{13}^{(i)} &= \tau_{13}^{i-1,i} + \left(\tau_{13}^{i,i+1} - \tau_{13}^{i-1,i}\right)\frac{\left(x_3 - x_3^{i-1,i}\right)}{h^{(i)}} \\ \tau_{23}^{(i)} &= \tau_{23}^{i-1,i} + \left(\tau_{23}^{i,i+1} - \tau_{23}^{i-1,i}\right)\frac{\left(x_3 - x_3^{i-1,i}\right)}{h^{(i)}} \end{aligned} \tag{4}$$

where $\tau_{13}^{i,i+1}$ is the shear stress at the interface between the i^{th} and $i+1^{th}$ plies, and $x_3^{i-1,i}$ is the value of the x_3 coordinate at the interface between the i-1^{th} and i^{th} laminae. Crucial to the analysis is finding averaged displacements in the laminate for a given in-plane loading condition. A thickness average of the mechanical parameters is defined as

$$\hat{\varphi} = \frac{1}{h}\int_h \varphi dx_3 \tag{5}$$

where *hat* denotes an averaged quantity and h represents the thickness over which the average is taken; it may be the thickness of the laminate or the thickness of one lamina. Then, the overall reduced stiffness properties can be obtained applying unit normal and shear loads and calculating the induced deformations. The analysis begins writing (a) the constitutive and (b) equilibrium equations in terms of averaged quantities. Damage in the form of cracks is analyzed as being discrete (not homogenized) with crack density λ_κ. Since the discrete nature of the cracks is included, the material between cracks is undamaged, with stiffness $\overline{Q}^{(k)}$ calculated in terms of undamaged (virgin) moduli [1,(5.23)] in the coordinates of lamina k (Figure 1). Then, the constitutive equations of the cracked lamina can be written in terms of averaged in-plane displacements and undamaged stiffness as follows

$$\begin{Bmatrix} \hat{\sigma}_1^{(k)} \\ \hat{\sigma}_2^{(k)} \\ \hat{\tau}_{12}^{(k)} \end{Bmatrix} = \bar{Q}^{(k)} \begin{Bmatrix} \hat{u}_{,1}^{(k)} \\ \hat{v}_{,2}^{(k)} \\ \left(\hat{u}_{,2}^{(k)} + \hat{v}_{,1}^{(k)}\right) \end{Bmatrix} \tag{6}$$

where *overline* denotes undamaged quantities, and $(\ )_{,1}$ and $(\ )_{,2}$ represent partial derivatives respect x_1 and x_2 directions, respectively. The remaining laminae have reduced properties that can be calculated in terms of their previously calculated damage values $D_2^{(m)}, D_6^{(m)}$ as follows

$$\begin{Bmatrix} \hat{\sigma}_1^{(m)} \\ \hat{\sigma}_2^{(m)} \\ \hat{\tau}_{12}^{(m)} \end{Bmatrix} = Q^{(m)} \begin{Bmatrix} \hat{u}_{,1}^{(m)} \\ \hat{v}_{,2}^{(m)} \\ \left(\hat{u}_{,2}^{(m)} + \hat{v}_{,1}^{(m)}\right) \end{Bmatrix} \tag{7}$$

where the damaged stiffness in the coordinate system of lamina *k* (Figure 1) is

$$Q^{(m)} = \left[T(\theta)\right]^{-1} \begin{bmatrix} \bar{Q}_{11}^{(m)} & (1-D_2^{(m)})\bar{Q}_{12}^{(m)} & 0 \\ (1-D_2^{(m)})\bar{Q}_{12}^{(m)} & \left(1-D_2^{(m)}\right)\bar{Q}_{22}^{(m)} & 0 \\ 0 & 0 & \left(1-D_6^{(m)}\right)\bar{Q}_{66}^{(m)} \end{bmatrix} \left[T(\theta)\right]^{-T} \tag{8}$$

where *k* and *m* are labels for the cracked lamina and the remaining laminae, respectively; $\left[T\right]^{-1}$ is the stress transformation matrix [1,(5.33)] from the material coordinate system of lamina *m* to lamina *k*, with the angle θ measured from *k* to *m* [1, Fig. 5.9], and $D_2^{(m)}$ and $D_6^{(m)}$ are variables which represent the transverse and shear stiffness reduction of the laminae $m \neq k$ [16, section 8.2]. Note in (8) that damage in the homogenized laminae *m* affects the shear-extension coupling terms with the implication that a balanced laminate might not remain balanced as damage evolves, perhaps differently, in various laminae.

Composites Part B, 41:124—132, 2010.

The constitutive equations for out-of-plane shear strains and stresses can be expressed in terms of interface shear stresses and averaged displacements by taking a weighted average of these equations (see Appendix). These equations are usually called shear lag equations [37], and they are written as follows

$$\begin{aligned}
\begin{Bmatrix} \hat{u}^{(i)}-\hat{u}^{(i-1)} \\ \hat{v}^{(i)}-\hat{v}^{(i-1)} \end{Bmatrix} &= \frac{h^{(i-1)}}{6}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i-1)} \begin{Bmatrix} \tau_{23}^{i-2,i-1} \\ \tau_{13}^{i-2,i-1} \end{Bmatrix} \\
&+\left(\frac{h^{(i-1)}}{3}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i-1)} + \frac{h^{(i)}}{3}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i)} \right)\begin{Bmatrix} \tau_{23}^{i-1,i} \\ \tau_{13}^{i-1,i} \end{Bmatrix} + \frac{h^{(i)}}{6}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i)} \begin{Bmatrix} \tau_{23}^{i,i+1} \\ \tau_{13}^{i,i+1} \end{Bmatrix}
\end{aligned} \tag{9}$$

Finally, the equilibrium equations in the *x* and *y* direction can be written as

$$\begin{aligned}
\hat{\sigma}_{1,1}^{(i)} + \hat{\tau}_{12,2}^{(i)} + \left(\tau_{13}^{i,i+1} - \tau_{13}^{i-1,i}\right)\Big/ h^{(i)} = 0 \\
\hat{\tau}_{12,1}^{(i)} + \hat{\sigma}_{2,2}^{(i)} + \left(\tau_{23}^{i,i+1} - \tau_{23}^{i-1,i}\right)\Big/ h^{(i)} = 0
\end{aligned} \tag{10}$$

Replacing equations (6)-(8) and (9) into equations (10) leads to a second-order system of partial differential equations for the unknown displacements. A solution for the second-order system of partial differential equations (PDE) (10) is found by the inverse method. That is, a solution like the one below (11) is proposed and verified to satisfy the PDE. Based on our familiarity with the direct solution of a much simpler problem, i.e., the $[0_n/90]_S$ laminate in [44,41], and prior use of the indirect method in [38,3,4,28], the following expression for the displacement is *proposed* herein as a solution for the system (10), as follows

$$\begin{Bmatrix} \hat{u}^{(1)} \\ \hat{u}^{(2)} \\ \vdots \\ \hat{u}^{(n)} \\ \hat{v}^{(1)} \\ \hat{v}^{(2)} \\ \vdots \\ \hat{v}^{(n)} \end{Bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix} \sinh(\lambda x_2) + \begin{bmatrix} 1/2\,\gamma_{12}^c \\ 1/2\,\gamma_{12}^c \\ \vdots \\ 1/2\,\gamma_{12}^c \\ \varepsilon_2^c \\ \varepsilon_2^c \\ \vdots \\ \varepsilon_2^c \end{bmatrix} x_2 + \begin{bmatrix} \varepsilon_1^c \\ \varepsilon_1^c \\ \vdots \\ \varepsilon_1^c \\ 1/2\,\gamma_{12}^c \\ 1/2\,\gamma_{12}^c \\ \vdots \\ 1/2\,\gamma_{12}^c \end{bmatrix} x_1 \tag{11}$$

where n is half the number of plies in the laminate; a_i and λ are unknown constants and ε_1^c, ε_2^c, γ_{12}^c are constant deformations that appear during the integration procedure. Replacing (11) into (10) leads to an eigenvalue system where a_i and λ are the eigenvectors and eigenvalues for the system. The final solution is the linear combination of all particular solutions, as follows

$$\begin{Bmatrix} \hat{u}^{(1)} \\ \hat{u}^{(2)} \\ \vdots \\ \hat{u}^{(n)} \\ \hat{v}^{(1)} \\ \hat{v}^{(2)} \\ \vdots \\ \hat{v}^{(n)} \end{Bmatrix} = \sum_{j=1}^{2n} A_j \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \\ a_{n+1} \\ a_{n+2} \\ \vdots \\ a_{2n} \end{bmatrix}_j \sinh(\lambda_j x_2) + \begin{bmatrix} 1/2\,\gamma_{12}^c \\ 1/2\,\gamma_{12}^c \\ \vdots \\ 1/2\,\gamma_{12}^c \\ \varepsilon_2^c \\ \varepsilon_2^c \\ \vdots \\ \varepsilon_2^c \end{bmatrix} x_2 + \begin{bmatrix} \varepsilon_1^c \\ \varepsilon_1^c \\ \vdots \\ \varepsilon_1^c \\ 1/2\,\gamma_{12}^c \\ 1/2\,\gamma_{12}^c \\ \vdots \\ 1/2\,\gamma_{12}^c \end{bmatrix} x_1 \tag{12}$$

where A_j, ε_1^c, ε_2^c, γ_{12}^c are unknown quantities. However, two λ_i eigenvalues are equal to zero, so the number of unknowns is reduced to $2n$+1.

To find the values of these constants, the following boundary conditions are used. First, stress-free conditions are assumed at the surfaces of the cracks in the cracked lamina k, as follows

$$\begin{aligned} \frac{1}{2l}\int_{-0.5}^{0.5} \hat{\sigma}_2^{(k)} dx_1 = 0, \quad \text{at } x_2 = \pm l \\ \frac{1}{2l}\int_{-0.5}^{0.5} \hat{\tau}_{12}^{(k)} dx_1 = 0, \quad \text{at } x_2 = \pm l \end{aligned} \tag{13}$$

Equation (13) represents a set of two independent conditions. Second, the stresses in the uncracked surfaces are related to the external loads by

$$\begin{aligned}
&\sum_{m=1}^{n}(1-\delta_{mk})h^{(m)}\int_{-0.5}^{0.5}\hat{\sigma}_2^{(m)}dx_1 = h\hat{\sigma}_2, \quad \text{at } x_2=\pm l\\
&\sum_{m=1}^{n}(1-\delta_{mk})h^{(m)}\int_{-0.5}^{0.5}\hat{\tau}_{12}^{(m)}dx_1 = h\hat{\tau}_{12}, \quad \text{at } x_2=\pm l\\
&\sum_{i=1}^{n}\frac{h^{(i)}}{2l}\int_{-l}^{l}\hat{\sigma}_1^{(i)}dx_2 = h\hat{\sigma}_1, \quad \text{at } x_2=\pm l
\end{aligned} \tag{14}$$

where $\delta_{mk}=1$, if $m=k$, 0 otherwise, and $\hat{\sigma}_1,\hat{\sigma}_2,\hat{\tau}_{12}$ are the components of the external load applied to the laminate in the material coordinates. Three independent conditions can be obtained from (14). Finally, the displacements in the x_2 direction are assumed to be the same for all un-cracked ply surfaces

$$\begin{aligned}
&\hat{u}^{(m)}=\hat{u}^{(r)}, \quad \text{at } x_2=\pm l \ \forall m\neq k\\
&\hat{v}^{(m)}=\hat{v}^{(r)}, \quad \text{at } x_2=\pm l \ \forall m\neq k
\end{aligned} \tag{15}$$

where r is the un-cracked lamina taken as reference. A total of $2(n-2)$ independent conditions that can be obtained from (15). Equations (13)-(15) represent a set of $2n+1$ boundary conditions, which allows us to calculate all the unknowns. Then, the average strains can be calculated as

$$\begin{aligned}
&\hat{\varepsilon}_1=\varepsilon_1^c\\
&\hat{\varepsilon}_2=\frac{1}{2l}\int_{-l}^{l}\hat{v}_{,2}^{(m)}\,dx_2\\
&\hat{\gamma}_{12}=\frac{1}{2l}\int_{-l}^{l}\left(\hat{u}_{,2}^{(m)}+\hat{v}_{,1}^{(m)}\right)dx_2
\end{aligned} \tag{16}$$

where m is one of the un-cracked laminae ($m\neq k$). To obtain the laminate compliance S in the material coordinate system of the cracked lamina, three unit-load cases are considered

$$\begin{Bmatrix}\hat{\sigma}_1\\ \hat{\sigma}_2\\ \hat{\tau}_{12}\end{Bmatrix}^{(a)}=\begin{bmatrix}1\\0\\0\end{bmatrix};\quad \begin{Bmatrix}\hat{\sigma}_1\\ \hat{\sigma}_2\\ \hat{\tau}_{12}\end{Bmatrix}^{(b)}=\begin{bmatrix}0\\1\\0\end{bmatrix};\quad \begin{Bmatrix}\hat{\sigma}_1\\ \hat{\sigma}_2\\ \hat{\tau}_{12}\end{Bmatrix}^{(c)}=\begin{bmatrix}0\\0\\1\end{bmatrix} \tag{17}$$

and the deformations obtained for each case are the components of S in the material coordinate system of the cracked lamina, as

Composites Part B, 41:124—132, 2010.

$$Q^{-1}(\lambda) = S(\lambda) = \left[\begin{Bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\gamma}_{12} \end{Bmatrix}^{(a)} \quad \begin{Bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\gamma}_{12} \end{Bmatrix}^{(b)} \quad \begin{Bmatrix} \hat{\varepsilon}_1 \\ \hat{\varepsilon}_2 \\ \hat{\gamma}_{12} \end{Bmatrix}^{(c)} \right] \tag{18}$$

The compliance matrix $\mathring{S}$ in the global coordinate system is obtained using the coordinate transformation

$$\left[\mathring{S}\right] = \left[T(\theta_k)\right]^{-1}\left[S\right]\left[T(\theta_k)\right]^{-T} \tag{19}$$

where [T] is the coordinate transformation matrix [1,(5.35)] and θ_k is the fiber direction of the cracked lamina. The overall elastic properties for the laminate can be written [1,(6.32)] as

$$E_X = \mathring{S}_{11}^{-1}; \quad E_Y = \mathring{S}_{22}^{-1}; \quad \nu_{XY} = -\mathring{S}_{12} / \mathring{S}_{11}; \quad G_{XY} = \mathring{S}_{33}^{-1} \tag{20}$$

A very important characteristic of the procedure presented here is that stiffness of the damaged lamina are function only of the undamaged elastic properties and the crack densities λ_k. Unlike plasticity-inspired formulations [16,Chapter 8], there is no need for defining damage evolution functions in terms of additional parameters, and thus no need to adjust such parameters using additional experimental data.

The damage parameters $D_2^{(k)}, D_6^{(k)}$, for lamina k can be calculated considering the reduction of the laminate stiffness due to matrix cracking in this lamina. The undamaged stiffness matrix of the laminate $\overline{Q}$ in the material coordinate system of the cracked lamina is calculated in terms of the undamaged properties for the laminae as follows

$$\overline{Q} = \overline{Q}^{(k)} \frac{h^{(k)}}{H} + \sum_{m}^{n-1} \overline{Q}^{(m)} \frac{h^{(m)}}{H} \tag{21}$$

where H is the laminate thickness, $\overline{Q}^{(k)}$ is the undamaged stiffness matrix of lamina k in the material coordinate system of lamina k; and $\overline{Q}^{(m)}$ are the undamaged stiffness matrices for the homogenized laminae $m \neq k$ in the material coordinate system of lamina *k*. When the crack density grows, the damaged laminate stiffness reduces according to

$$Q = Q^{(k)} \frac{h^{(k)}}{H} + \sum_{m}^{n-1} Q^{(m)} \frac{h^{(m)}}{H} \tag{22}$$

Since the damaged laminate stiffness $Q = S^{-1}$ can be computed from (18), and the damaged lamina stiffness $Q^{(m)}$ are given by the coefficients in (8), then, $Q^{(k)}$ can be computed from (22). Finally, the damage variables are calculated as follows

$$\begin{aligned} D_2^{(k)}(\lambda_k) &= 1 - Q_{22}^{(k)} / \bar{Q}_{22}^{(k)} \\ D_6^{(k)}(\lambda_k) &= 1 - Q_{66}^{(k)} / \bar{Q}_{66}^{(k)} \end{aligned} \tag{23}$$

where (8) and (23) are consistent with each other [16, section 8.2, eqs. (8.62,8.63,8.67)].

2.2. Damage Activation Function

The energy release rate (ERR) associated with crack opening displacements in mode I and mode II can be expressed as

$$\begin{aligned} G_I &= -\frac{\partial U_I}{\partial A} \\ G_{II} &= -\frac{\partial U_{II}}{\partial A} \end{aligned} \tag{24}$$

where U_I and U_{II} are the strain energy for mode I and mode II, respectively; and *A* is the crack area. The strain energy components U_I and U_{II} can be expressed in terms of the stress and strain in the material coordinate system of the cracked lamina as

$$\begin{aligned} G_I &= -\frac{V_{uc}}{2} \frac{\partial \left\{ \hat{\sigma}_1 \hat{\varepsilon}_1 + \hat{\sigma}_2 \hat{\varepsilon}_2 \right\}}{\partial A} \\ G_{II} &= -\frac{V_{uc}}{2} \frac{\partial \left\{ \hat{\tau}_{12} \hat{\gamma}_{12} \right\}}{\partial A} \end{aligned} \tag{25}$$

where V_{uc} is the volume of the unit cell. The average stress in the laminate $\hat{\sigma}$ can be expressed in terms of the average strain $\hat{\varepsilon}$, using the laminate constitutive equations. Therefore, (25) can be rewritten as

$$\begin{aligned} G_I &= -\frac{V_{uc}}{2}\left\{\hat{\varepsilon}_1\left(\frac{\partial Q_{11}}{\partial A}\hat{\varepsilon}_1+\frac{\partial Q_{12}}{\partial A}\hat{\varepsilon}_2+\frac{\partial Q_{16}}{\partial A}\hat{\gamma}_{12}\right)+\varepsilon_2\left(\frac{\partial Q_{12}}{\partial A}\hat{\varepsilon}_1+\frac{\partial Q_{22}}{\partial A}\hat{\varepsilon}_2+\frac{\partial Q_{26}}{\partial A}\hat{\gamma}_{12}\right)\right\} \\ G_{II} &= -\frac{V_{uc}}{2}\hat{\gamma}_{12}\left(\frac{\partial Q_{16}}{\partial A}\hat{\varepsilon}_1+\frac{\partial Q_{26}}{\partial A}\hat{\varepsilon}_2+\frac{\partial Q_{66}}{\partial A}\hat{\gamma}_{12}\right) \end{aligned} \tag{26}$$

where $Q=S^{-1}$ is given by (18), $\frac{\partial}{\partial A}=\frac{\partial}{\partial \lambda}$, and A is the crack area in the Vuc; that is, four times the thickness of the lamina (2 faces of the crack times 2 symmetric cracking laminae). While several failure criteria in terms of ERR have been proposed for composite materials [43], the failure criterion proposed in [40] is recast here as a damage activation function [15, chapter 8] as follows

$$g(\lambda)=(1-r)\sqrt{\frac{G_I(\lambda)}{G_{IC}}}+r\frac{G_I(\lambda)}{G_{IC}}+\frac{G_{II}(\lambda)}{G_{IIC}}-1\leq 0 \tag{27}$$

where $r=G_{IC}/G_{IIC}$ and G_{IC} and G_{IIC} are the critical values of the ERR for mode I and II. Since the crack is assumed to span the entire lamina thickness (Fig. 1) and to propagate in the longitudinal direction, for the case of thick laminae having initial defects smaller than the lamina thickness [46], equation (27) may yield somewhat conservative estimates of the first crack strain, as small cracks need to propagate through the thickness before developing into thickness cracks.

3. SOLUTION ALGORITHM

The proposed algorithm consists of (a) strain steps, (b) laminate-iterations, and (c) lamina-iterations. The state variables for the laminate are the array of crack densities for all laminae λ_k and the strain tensor ε. At each load (strain) step, the strain on the laminate is increased and the laminae are checked for damage modes by evaluating the damage activation functions $g\leq 0$ of all possible modes of damage, including

1. Longitudinal tension
2. Longitudinal compression
3. Transverse tension
4. Transverse compression

Composites Part B, 41:124—132, 2010.

3.1. Lamina iterations

When matrix cracking is detected (mode 3) in, say, lamina *"k"*, an RMA is invoked to iterate and adjust the crack density λ_k in the k-lamina in such a way that g_k returns to zero while maintaining equilibrium. The iterative procedure works as follows. At a given strain level ε for the laminate and given λ_k for lamina *k*, calculate the value of the damage activation function g_k and the damage variables $D_2^{(k)}(\lambda_k), D_6^{(k)}(\lambda_k)$, both univocal functions of λ_k. The RMA calculates the increment (decrement) of crack density as [16, chapter 8],

$$\Delta\lambda_k = -g_k \,/\, \frac{\partial g_k}{\partial \lambda_k} \tag{28}$$

until $g_k = 0$ is satisfied within a given tolerance, for all $k=1...N$ where N is the number of laminae in the laminate. The analysis starts with a negligible value of crack density present in all laminae (λ=0.01 cracks/mm were used in the examples).

3.2. Laminate iterations

To calculate the stiffness reduction of a cracked lamina (*k*-lamina), all of the other laminae (*m*-laminae) in the laminate are considered undamaging during the course of lamina-iterations in lamina *k*, but with damaged properties calculated according to the current values of their damage variables $D_2^{(m)}, D_6^{(m)}$. Given a trial value of λ_k, the analytical solution provides $g_k, D_2^{(k)}$, and $D_6^{(k)}$ for lamina *k* assuming all other laminae $m \neq k$ do not damage while performing lamina iterations in lamina *k*. Since the solution for lamina *k* depends on the stiffness of the remaining laminae, a converged iteration for lamina *k* does not guarantee convergence for the same lamina once the damage in the remaining laminae is updated, unless of course the remaining laminae remain undamaged. In other words, within a given strain step, the stiffness and damage of all the laminae are interrelated and they must all converge. This can be accomplished by laminate-iterations; that is, looping over all laminae repeatedly until all laminae converge to $g_k = 0$ for all *k*. Unlike classical RMA set up for plasticity (where the hardening parameter is monotonically increasing [16, chapter 8]), the crack density λ_k must be allowed to decrease if in the course of laminate iterations, other laminae $m \neq k$ sustain additional damage that makes the laminate more compliant and thus requires a reduction of λ_k.

Table 1. Material properties for the laminates considered in study.

	IM600/#133 [28]	**Varna et al. [2]**
E_1 (GPa)	147	44.7
E_2 (GPa)	8.31	12.7
ν_{12}	0.352	0.297
G_{12} (GPa)	4.7	5.8
G_{23} (GPa)	2.865 *	4.5 *
Ply thickness (mm)	0.14	0.144
G_{IC} kJ/m^2 [43]	0.175	0.175
G_{IIC} kJ/m^2 [43]	1.561	1.561

* Assumed value.

Composites Part B, 41:124—132, 2010.

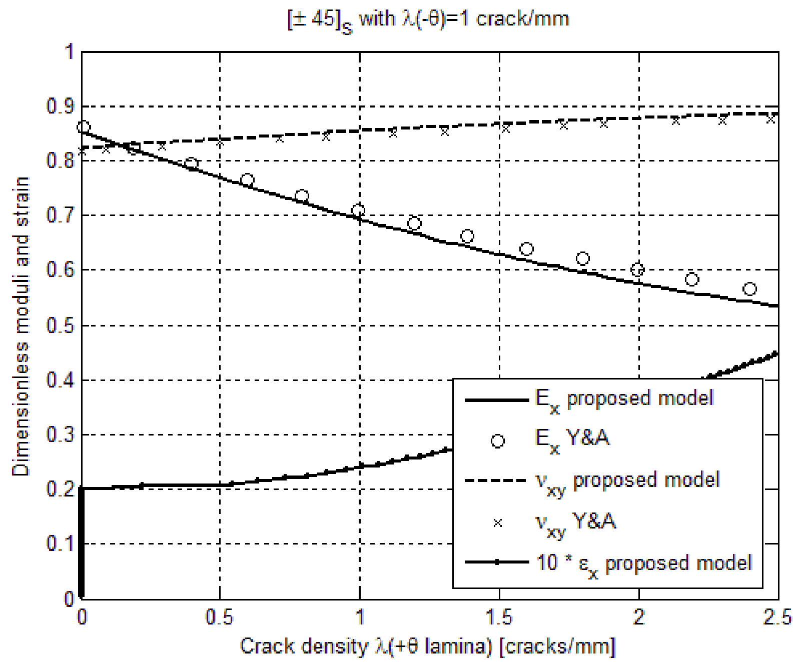

Figure 2. Predicted moduli and strain required to achieve increasing values of crack density. Moduli results compared with predictions from [28].

4. RESULTS AND DISCUSSION

Comparison between modulus $E_x / \bar{E}_x$ and Poisson's ratio ν_{xy} predicted by the proposed formulation and predictions from [28] is presented in Figure 2 to verify the calculation of reduced stiffness due to arbitrarily oriented matrix cracks. A $[\pm 45]_S$ laminate made of IM600/#133 composite with material properties given in Table 1 is used, as it was in [28]. The change of Young's modulus is obtained as function of the crack density λ_1 of the $\theta = 45^o$ lamina while the crack density λ_2 of the $\varphi = -45^o$ lamina was kept constant and equal to *1 crack/mm*, as it was in [28]. An excellent match is observed between the two formulations. While [28] computed the stiffness reduction in both laminae simultaneously using a formulation based on oblique coordinates, the proposed method computes the stiffness reduction one lamina at a time, in a Cartesian coordinate system, and uses an iterative method to converge to the solution for all laminae. The use of a Cartesian coordinate system is advantageous because methods based on oblique coordinates experience numerical instability when the angle between the two oblique axes, and thus between two damaging laminae, is either small or close to π [38]. The proposed formulation has no such limitations. Also note that the formulation in [28] does not provide the strain corresponding to crack density because it does not have a damage evolution criteria such as in the model proposed herein; the model in [28] is driven by crack density only. Also note that the strain profile displayed in Figure 2 is peculiar because of the physically unrealistic nature of this example, namely, the crack density was kept constant at λ=1 crack/mm.

A classical example of a laminate that display significant transverse matrix cracking is the cross ply; in this case a $[0/90_8/0_{1/2}]_S$, where the crack open in pure mode I. It can be seen in Figure 3 that the proposed formulation predicts well the strain required for crack initiation and the evolution of crack density with applied strain when compared to experimental data of [2]. Furthermore, the degradation of Young's modulus $E_x / \bar{E}_x$ and Poisson's ratio ν_{xy}, as shown in Figure 4, is predicted accurately as a function of crack density, thus verifying that the proposed formulation can predict accurately crack initiation, evolution, and stiffness reduction.

Deviating from the classical example, a $[0/70_4/-70_4/0_{1/2}]_S$ laminate is prone to matrix cracking but in this case the cracks are subjected to both mode I and mode II. It can be seen in Figure 5 that the proposed formulation predicts well the strain required for crack initiation and the evolution of crack density with applied strain when compared to experimental data of [2].

With even more preponderance of mode II crack opening displacement, a $[0/55_4/-55_4/0_{1/2}]_S$ laminate is prone to matrix cracking but in this case the laminate is much stiffer in the load direction. It can be seen in Figure 6 that the proposed formulation predicts well the strain required for crack initiation (experimental set 1), and the evolution of crack density with applied strain follows the trend of the experimental data quite well.

Damage initiation and degradation of laminate Young's modulus $E_x / \bar{E}_x$ and Poisson's ratio ν_{xy} are accurately predicted as shown in Figure 7 for a $[0_2/90_4]_S$ laminate. Load redistribution from the 90-deg lamina to the longitudinal laminae eventually brings about longitudinal tensile failure

Composites Part B, 41:124—132, 2010.

as clearly shown in the figure, which occurs when the stress in of the 0-deg lamina reaches the longitudinal tensile strength of the unidirectional (UD) lamina F_{1T} .

Excellent predictions are obtained for less stiff laminates as shown in Figures 8-10 for $[15/\text{-}15/90_4]_S$, $[30/\text{-}30/90_4]_S$, and $[40/\text{-}40/90_4]_S$ laminates, where models predictions are compared to experimental data from [23]. The proposed formulation yields good predictions even when the main load carrying laminae are at a steep angle, as it is the case for the $[40/\text{-}40/90_4]_S$ laminate.

The values of GIc and GIIc from [43] were determined experimentally at room temperature and used in this manuscript to analyze the experimental data of [2,23] that were also obtained at room temperature. In order to perform the analysis at temperatures other than room temperature it is necessary to extend the formulation by calculating residual thermal stresses, and this will be the subject of a subsequent study.

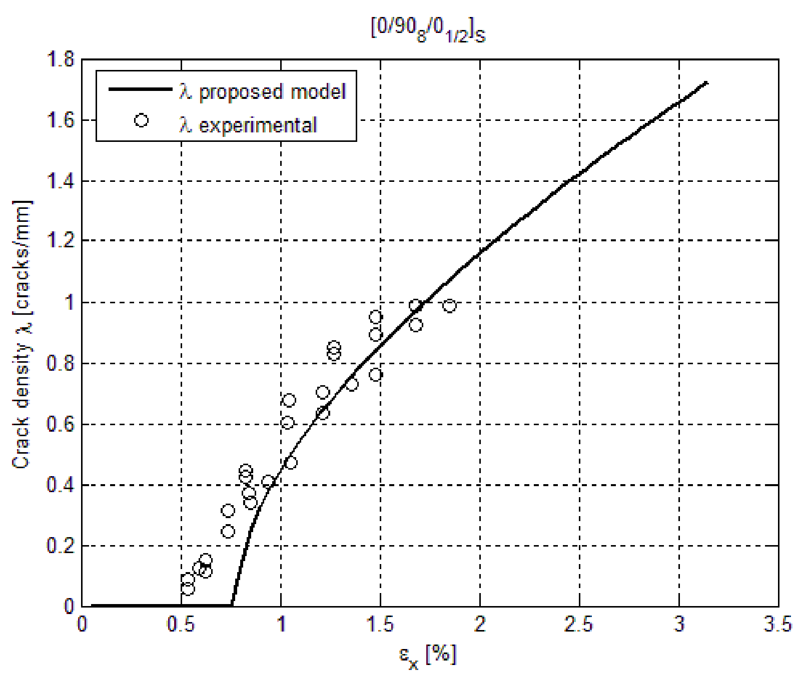

Figure 3. Crack density vs. applied strain for a $[0/90_8/0_{1/2}]_S$ laminate.

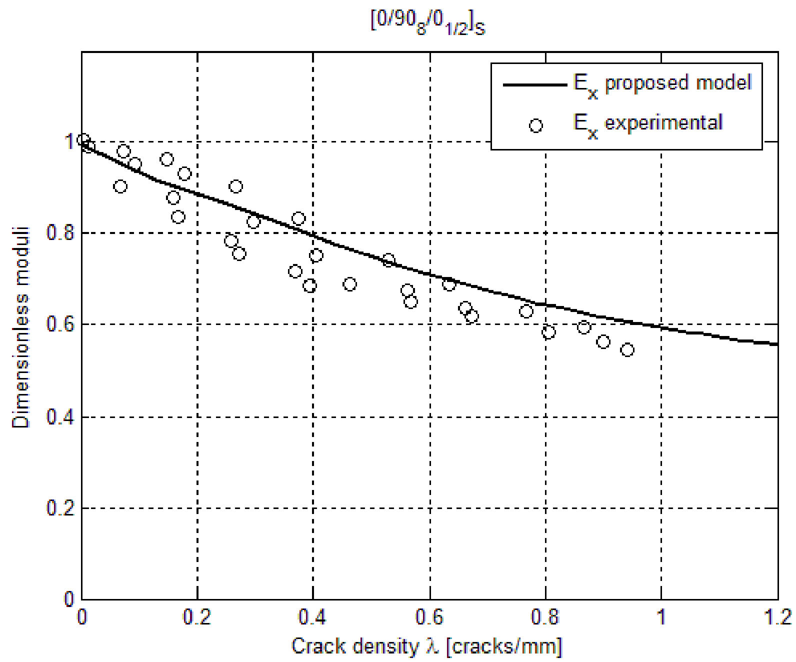

Figure 4. Laminate $E_x / \bar{E}_x$ and ν_{xy} vs. crack density for a $[0/90_8/0_{1/2}]_S$ laminate.

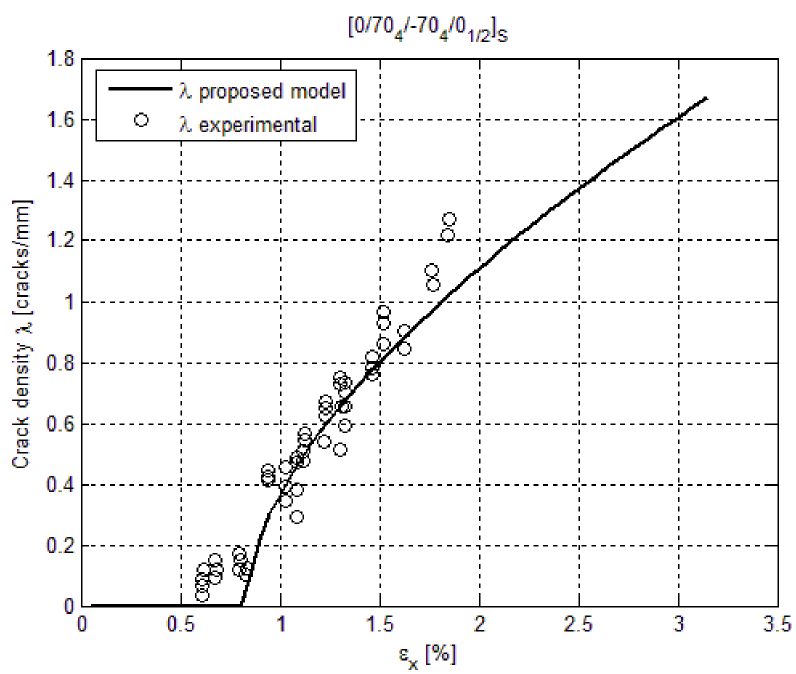

Figure 5. Crack density vs. applied strain for a $[0/70_4/\text{-}70_4/0_{1/2}]_S$ laminate.

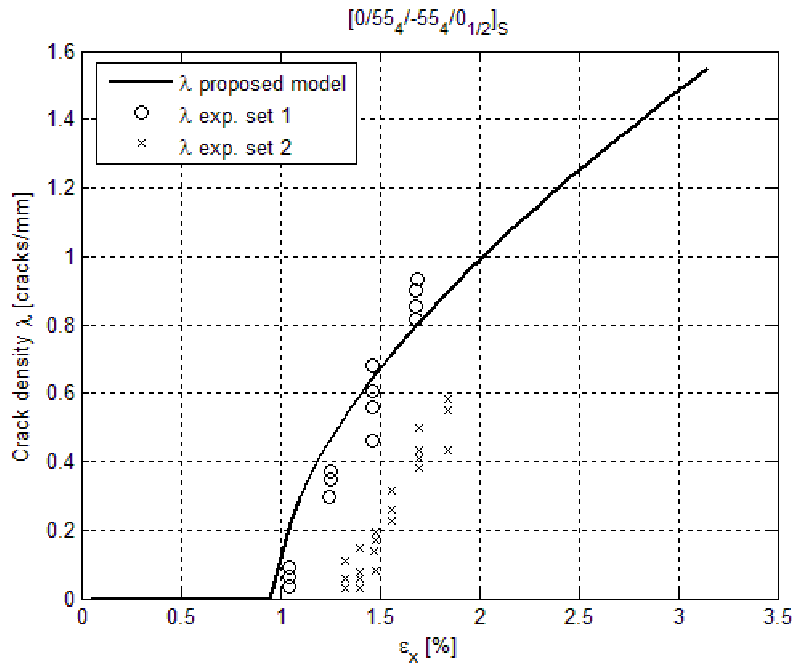

Figure 6. Crack density vs. applied strain for a [0/55$_4$/-55$_4$/0$_{1/2}$]$_S$ laminate.

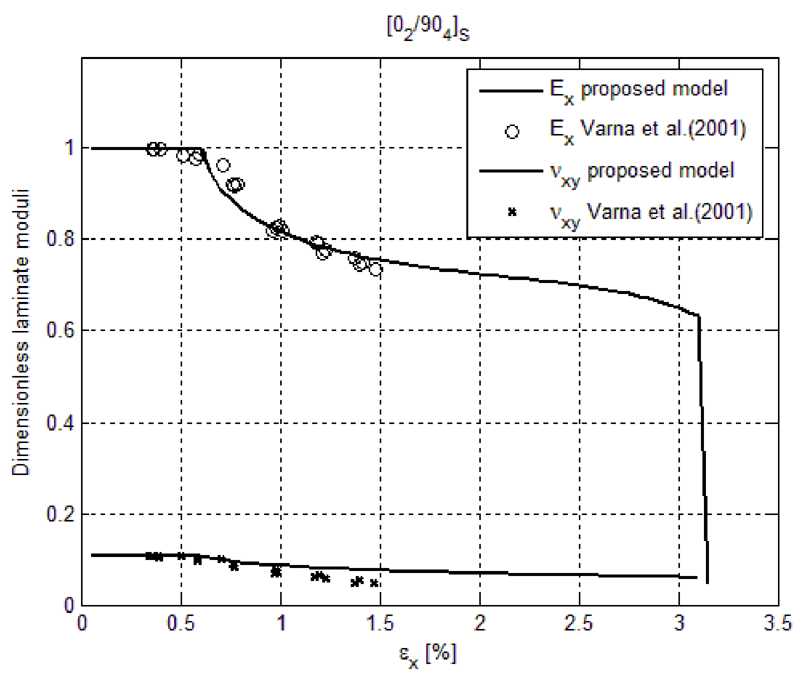

Figure 7. Loss of laminate modulus E_x / $\bar{E}_x$ and ν_{xy} as a function of applied strain for a cross-ply laminate.

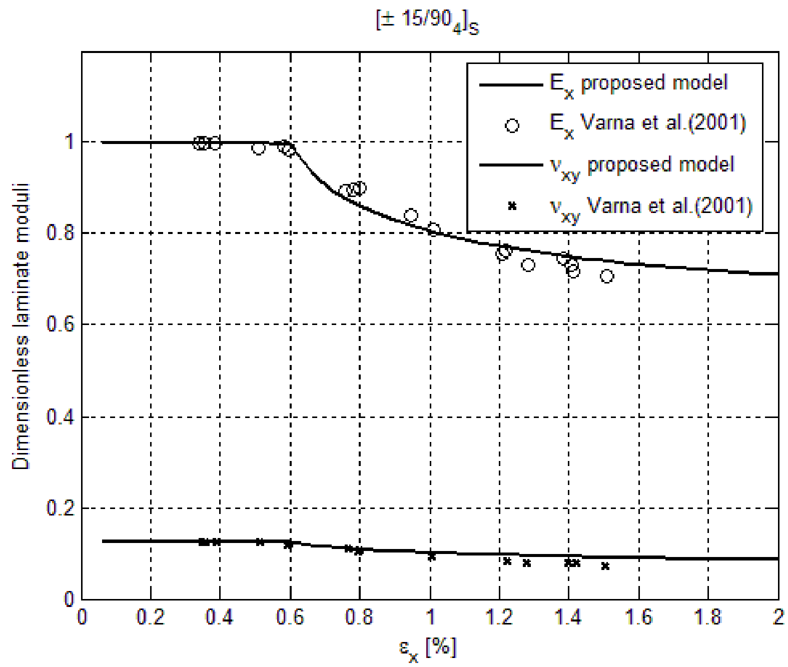

Figure 8. **Loss of laminate modulus modulus $E_x / \bar{E}_x$ and ν_{xy} as a function of applied strain for a [15/-15/90$_4$]$_S$ laminate.**

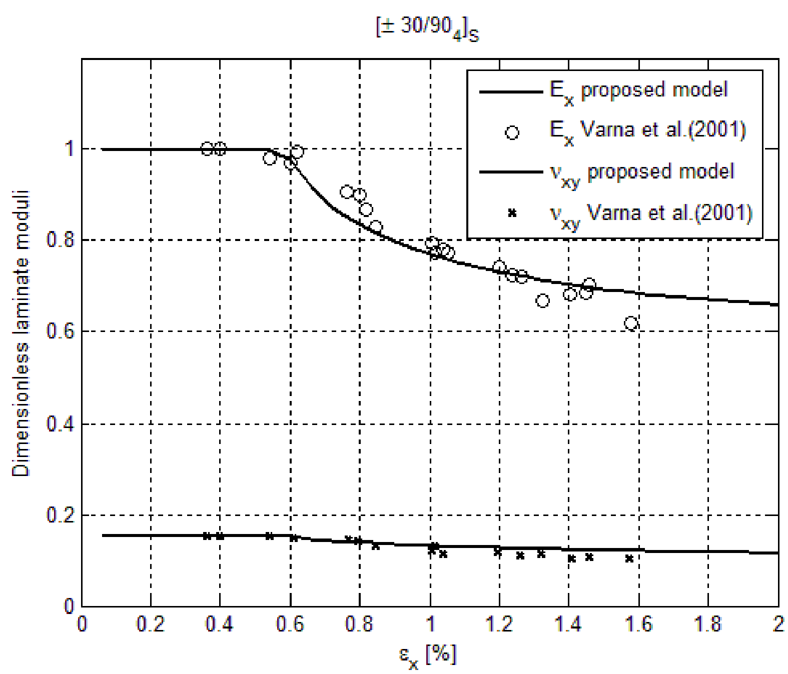

Figure 9. **Loss of laminate modulus modulus $E_x / \bar{E}_x$ and ν_{xy} as a function of applied strain for a [30/-30/90$_4$]$_S$ laminate.**

Composites Part B, 41:124—132, 2010.

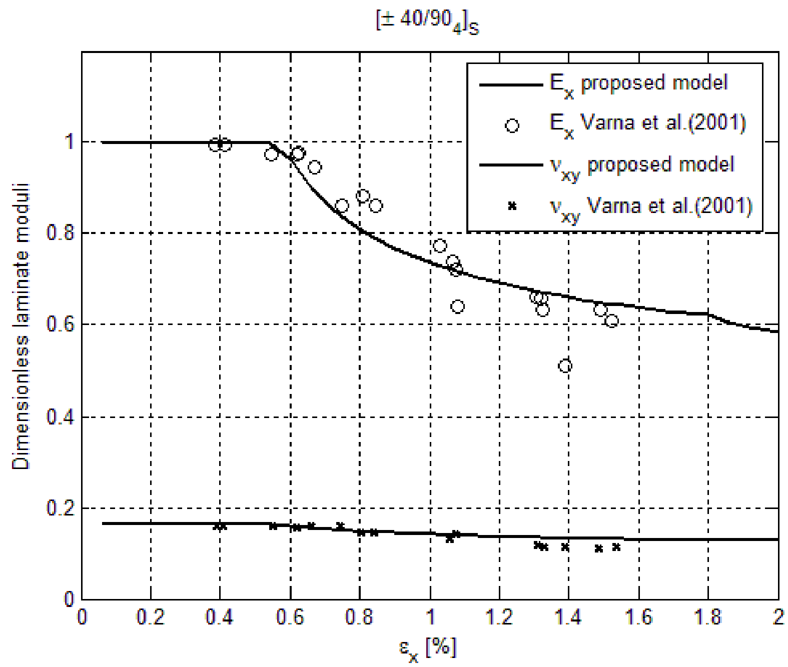

Figure 10. **Loss of laminate modulus modulus $E_x / \bar{E}_x$ and ν_{xy} as a function of applied strain for a [40/-40/90$_4$]$_S$ laminate.**

5. CONCLUSIONS

Good agreement between predicted results and experimental data for a variety of laminates support the thesis that critical energy release rate (ERR) in mode I and II can be used as material properties, independent of lamina thickness and LSS, to predict crack initiation in lieu of strength failure criteria. Based on just two experimental values of critical ERR (mode I and II), without postulating evolution (kinetic) equations and without the associated experimental effort needed to determine their adjustable parameters, the proposed model is able to predict accurately the strain at crack initiation, the evolution of crack density as a function of applied strain and the reduction in laminate moduli. This is possible because the proposed model provides analytical, accurate prediction of stiffness degradation as a function of crack density in each lamina. The proposed model is focused on matrix-dominated damage but it proves crucial for accurate prediction of fiber-dominated modes as well, because the stress redistribution resulting from matrix modes is critical for accurate prediction of fracture due to fiber dominated modes. Existing fiber-dominated failure models, e.g., longitudinal tensile failure based on Weibull distribution of fiber strength and longitudinal compression failure based on Gaussian distribution of fiber misalignment, are able to accurately predict fiber-failure modes provided the stress in the fiber direction is accurately determined taking into account stress redistribution from the cracking laminae. The predicted stress redistribution, from the cracking lamina to the rest of the laminate, given by the proposed model is accurate, as it is shown by accurate prediction of longitudinal failure of the 0-deg lamina in a $[0_2/90_4]_S$ laminate after the 90-deg lamina is degraded by matrix cracks. Intermediate damage variables D_2, D_6, for the laminae (both univocal functions of the crack densities in the laminae, which are the only state variables), and a modified return mapping algorithm are shown to be effective in expanding the shear-lag solution for one cracking lamina to situations where several, if not all laminae might be degrading. In this way, the proposed formulation predicts fracture initiation, evolution, stiffness degradation, and stress redistribution accurately in terms of measurable material properties (UD lamina moduli and critical ERR). Unlike plasticity-inspired CDM models, the proposed model does not need additional parameters that would have to be determined with additional experimentation at significant cost. Comparison with available experimental data for crack initiation and evolution, as well as for laminate stiffness degradation is good.

APPENDIX

The relationship between the out-of-plane shear stresses and the averaged displacements can be obtained by calculating a weighted average of in-plane deformation. Using Hooke's law, the shear strains can be expressed in terms of the shear stresses as

$$\begin{aligned} \gamma_{23}^{(i)} &= S_{44}^{(i)}\tau_{23}^{(i)} + S_{45}^{(i)}\tau_{13}^{(i)} \\ \gamma_{13}^{(i)} &= S_{45}^{(i)}\tau_{23}^{(i)} + S_{55}^{(i)}\tau_{13}^{(i)} \end{aligned} \quad . \tag{A1}$$

Equation (A1) can be written in terms of displacements, and making use of (3)-(4), the following equation is obtained

$$\begin{aligned} u_{,3}^{(i)} + w_{,1}^{(i)} &= S_{45}^{(i)}\left[\tau_{23}^{i,i+1} + \left(\tau_{23}^{i-1,i} - \tau_{23}^{i,i+1}\right)\frac{h_i - x_3}{h^{(i)}}\right] + S_{55}^{(i)}\left[\tau_{13}^{i,i+1} + \left(\tau_{13}^{i-1,i} - \tau_{13}^{i,i+1}\right)\frac{h_i - x_3}{h^{(i)}}\right] \\ v_{,3}^{(i)} + w_{,2}^{(i)} &= S_{44}^{(i)}\left[\tau_{23}^{i,i+1} + \left(\tau_{23}^{i-1,i} - \tau_{23}^{i,i+1}\right)\frac{h_i - x_3}{h^{(i)}}\right] + S_{45}^{(i)}\left[\tau_{13}^{i,i+1} + \left(\tau_{13}^{i-1,i} - \tau_{13}^{i,i+1}\right)\frac{h_i - x_3}{h^{(i)}}\right] \end{aligned} \tag{A2}$$

The following weighted average is defined

Composites Part B, 41:124—132, 2010.

$$\frac{1}{h^{(i)}}\int_{h_{i-1}}^{h_i}(\quad)\left(h_i - x_3\right)dx_3 \tag{A3}$$

Applying the operator shown in (A3) on both sides of (A2), the following relationship is obtained

$$\begin{aligned}
-u(h_{i-1}) + \hat{u}^{(i)} &= \frac{h^{(i)}}{6}S_{45}^{(i)}\tau_{23}^{i,i+1} + \frac{h^{(i)}}{3}S_{45}^{(i)}\tau_{23}^{i-1,i} + \frac{h^{(i)}}{6}S_{55}^{(i)}\tau_{13}^{i,i+1} + \frac{h^{(i)}}{3}S_{55}^{(i)}\tau_{13}^{i-1,i} \\
-v(h_{i-1}) + \hat{v}^{(i)} &= \frac{h^{(i)}}{6}S_{44}^{(i)}\tau_{23}^{i,i+1} + \frac{h^{(i)}}{3}S_{44}^{(i)}\tau_{23}^{i-1,i} + \frac{h^{(i)}}{6}S_{45}^{(i)}\tau_{13}^{i,i+1} + \frac{h^{(i)}}{3}S_{45}^{(i)}\tau_{13}^{i-1,i}
\end{aligned} \tag{A4}$$

where $u(h_{i\text{-}1})$ and $v(h_{i\text{-}1})$ are the in-plane displacements evaluated at $z = h_{i\text{-}1}$. Following a similar procedure for ply (i-1), but using $(z - h_{i\text{-}2})$ as a weight, the following relation is obtained

$$\begin{aligned}
u(h_{i-1}) + \hat{u}^{(i-1)} &= \frac{h^{(i-1)}}{6}S_{45}^{(i-1)}\tau_{23}^{i-2,i-1} + \frac{h^{(i-1)}}{3}S_{45}^{(i-1)}\tau_{23}^{i-1,i} + \frac{h^{(i-1)}}{6}S_{55}^{(i-1)}\tau_{13}^{i-2,i-1} + \frac{h^{(i-1)}}{3}S_{55}^{(i-1)}\tau_{13}^{i-1,i} \\
v(h_{i-1}) + \hat{v}^{(i-1)} &= \frac{h^{(i-1)}}{6}S_{44}^{(i-1)}\tau_{23}^{i-2,i-1} + \frac{h^{(i-1)}}{3}S_{44}^{(i-1)}\tau_{23}^{i-1,i} + \frac{h^{(i-1)}}{6}S_{45}^{(i-1)}\tau_{13}^{i-2,i-1} + \frac{h^{(i-1)}}{3}S_{45}^{(i-1)}\tau_{13}^{i-1,i}
\end{aligned} \tag{A5}$$

Adding (A4) and (A5) and presenting the result in a matrix form we obtain

$$\begin{aligned}
\begin{Bmatrix} \hat{u}^{(i)} - \hat{u}^{(i-1)} \\ \hat{v}^{(i)} - \hat{v}^{(i-1)} \end{Bmatrix} = {} & \frac{h^{(i-1)}}{6}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i-1)} \begin{Bmatrix} \tau_{23}^{i-2,i-1} \\ \tau_{13}^{i-2,i-1} \end{Bmatrix} + \left(\frac{h^{(i-1)}}{3}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i-1)} + \frac{h^{(i)}}{3}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i)} \right) \begin{Bmatrix} \tau_{23}^{i-1,i} \\ \tau_{13}^{i-1,i} \end{Bmatrix} \\
& \frac{h^{(i)}}{6}\begin{bmatrix} S_{45} & S_{55} \\ S_{44} & S_{45} \end{bmatrix}^{(i)} \begin{Bmatrix} \tau_{23}^{i,i+1} \\ \tau_{13}^{i,i+1} \end{Bmatrix}
\end{aligned} \tag{A6}$$

Composites Part B, 41:124—132, 2010.

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