Composite Beam Element with Layerwise Plane Sections

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Abstract: Based on generalized laminate plate theory (GLPT), the formulation of a one-dimensional laminated beam finite element with layerwise constant shear (BLCS) is presented. BLCS formulation is equivalent to a first-order shear deformation beam theory (Timoshenko beam theory) on each layer, and a cross section of the beam therefore does not necessarily remain plane through the laminate but only through each layer. Plane stress is assumed through both the thickness and width of the beam in the constitutive equation for a lamina. Details are presented for transforming the layerwise constant shear stresses obtained from constitutive relations into parabolic shear stress distributions. The layerwise representation of in-plane displacement through the thickness results in the formulation of a relatively simple beam element. Numerical analyses are presented for a three-node BLCS element integrated with two Gauss points. The accuracy of the element is evaluated by comparing the predictions to elasticity and experimental results.

INTRODUCTION

Bernoulli-Euler (classical) and Timoshenko beam theories assume that originally plane cross sections remain plane after deformation. This assumption is the basis for the first-order shear deformation theory (FSDT), and is sufficiently accurate for isotropic beams and for layered composite beams with plies of similar stiffnesses, but it leads to serious discrepancies with the actual state of stresses in laminated beams when one or more layers have quite different mechanical properties. Reddy (1987) presented a generalized laminate plate theory (GLPT) from which a number of particular theories, including FSDT, can be derived. If in-plane displacements are layerwise linear through the thickness, a constant interlaminar shear on each layer is obtained. This is equivalent to using Timoshenko beam theory on each layer, where the in-plane displacements of the laminate are expressed layer-wise, and the shear stresses are constant through the thickness of each layer. Yuan and Miller (1989) developed an N-layer beam element with constant shear strain on each layer. A thorough discussion of shear deformation theories is presented by Noor and Burton (1989) and Kapania and Raciti (1989). Chaudhuri and Seide (1987) presented a method to compute the interlaminar shear stresses for a quadratic laminated triangular element. Similarly, Reddy et al. (1989) described a layerwise computation of shear stresses for a laminated plate.

A number of refined but complex theories for laminated beams are available. Based on GLPT, we present in this paper the development of a one-dimensional beam finite element with layerwise constant shear (BLCS). As Reddy et al. (1989) show, the assumption of layerwise linear in-plate displacements simplifies the element formulation without compromising its accuracy. The model can predict the linear elastic behavior of straight cross-ply–laminated beams with rectangular cross sections consisting of symmetric or asymmetric laminates. The layerwise constant shear stresses are transformed into parabolic distributions.

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BLCS FORMULATION

Based on transverse incompressibility and layerwise representation of in-plane displacement (see Reddy et al. 1989), the displacements of a point in a laminated beam can be written as

$$u_1(x, z) = u(x) + U(x, z), \qquad u_2(x, z) = w(x) \tag{1a,b}$$

where u and w = the longitudinal and transverse displacements of a point on the reference axis of the laminate, respectively; and $U(x, z)$ = layerwise in-plane displacement that can be approximated linearly by interpolation functions. The transformed stress-strain relation of an orthotropic lamina under the assumption of plane stress in the x-y plane and without the transverse normal stress component (see Jones 1975) can be written as

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_{xy} \\ \sigma_{yz} \\ \sigma_{xz} \end{Bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & \bar{Q}_{16} & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & \bar{Q}_{26} & 0 & 0 \\ \bar{Q}_{16} & \bar{Q}_{26} & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & \bar{Q}_{45} \\ 0 & 0 & 0 & \bar{Q}_{45} & \bar{Q}_{55} \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{Bmatrix} \tag{2}$$

where $\bar{Q}_{ij}$ = the transformed reduced stiffnesses. To represent the state of stress in each lamina, the following approximations are used: $\sigma_y = \sigma_{yz} = 0$ and, for a cross-ply lamina, $\sigma_{xy} = 0$. Imposing these conditions in (2), we obtain

$$\sigma_x = E_x\varepsilon_x, \qquad \sigma_{xz} = G_{xz}\gamma_{xz} \tag{3a,b}$$

where

$$E_x = \bar{Q}_{11} + \bar{Q}_{12}\frac{\bar{Q}_{16}\bar{Q}_{26} - \bar{Q}_{12}\bar{Q}_{66}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}\bar{Q}_{26}} + \bar{Q}_{16}\frac{\bar{Q}_{12}\bar{Q}_{26} - \bar{Q}_{16}\bar{Q}_{22}}{\bar{Q}_{22}\bar{Q}_{66} - \bar{Q}_{26}\bar{Q}_{26}} \tag{4a}$$

$$G_{xz} = -\frac{\bar{Q}_{45}^2}{\bar{Q}_{44}} + \bar{Q}_{55} \tag{4b}$$

The constitutive equation of the laminate can be derived by following an approach similar to Reddy et al. (1989). In the finite element formulation of BLCS, the strain-displacement relation is defined as

$$\boldsymbol{\varepsilon}^o = \mathbf{B}_L\boldsymbol{\Delta}^o, \qquad (\boldsymbol{\Delta}^o)^T = (u_1w_1 \cdots u_mw_m) \tag{5a}$$

$$\boldsymbol{\varepsilon}^j = \bar{\mathbf{B}}_L\boldsymbol{\Delta}^j, \qquad \boldsymbol{\Delta}^j = \mathbf{U}^j \tag{5b}$$

where the compatibility matrices $\mathbf{B}_L$ and $\bar{\mathbf{B}}_L$ for an m-node element can be expressed in terms of the interpolation functions H_i as

$$\mathbf{B}_L = \begin{bmatrix} \partial H_1/\partial x & 0 & \partial H_2/\partial x & 0 & \cdots & \partial H_m/\partial x & 0 \\ 0 & \partial H_1/\partial x & 0 & \partial H_2/\partial x & \cdots & 0 & \partial H_m/\partial x \end{bmatrix} \quad (6a)$$

$$\bar{\mathbf{B}}_L = \begin{bmatrix} \partial H_1/\partial x & \partial H_2/\partial x & \cdots & \partial H_m/\partial x \\ H_1 & H_2 & \cdots & H_m \end{bmatrix} \quad (6b)$$

Using (5*a*), (5*b*), and the constitutive equation of the laminate, and applying the principle of virtual work to the equilibrium equation, we obtain the element model as follows:

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_N \\ (\mathbf{B}_1)^T & \mathbf{D}_{11} & \mathbf{D}_{12} & \cdots & \mathbf{D}_{1N} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ (\mathbf{B}_N)^T & \mathbf{D}_{N1} & \mathbf{D}_{N2} & \cdots & \mathbf{D}_{NN} \end{bmatrix} \begin{bmatrix} \mathbf{\Delta}^0 \\ \mathbf{\Delta}^1 \\ \cdot \\ \mathbf{\Delta}^N \end{bmatrix} = \frac{1}{b} \begin{bmatrix} \mathbf{F} \\ \mathbf{F}_x^1 \\ \cdot \\ \mathbf{F}_x^N \end{bmatrix} \quad (7)$$

where b = the beam width; **F** includes transverse (f_z) and axial (f_x) force vectors applied at the reference axis; $\mathbf{F}_x^i$ contains axial (f_x^i) force vectors applied at the laminate interfaces; and the submatrices are defined as

$$\mathbf{A}_1 = \int_0^L (\mathbf{B}_L)^T \mathbf{A} \mathbf{B}_L \, dx \quad (8a)$$

$$\mathbf{B}_i = \int_0^L (\mathbf{B}_L)^T \mathbf{B}^i \bar{\mathbf{B}}_L \, dx \quad (8b)$$

$$\mathbf{D}_{ij} = \int_0^L (\bar{\mathbf{B}}_L)^T \mathbf{D}^{ij} \bar{\mathbf{B}}_L \, dx \quad (8c)$$

Since the two-node linear element can only model constant moment, a three-node quadratic isoparametric element is used in this study. Also, to avoid "shear-locking," two-point Gauss integration along the reference axis is adopted (see Cook et al. 1989).

COMPUTATION OF PARABOLIC SHEAR STRESS

The constant shear stress $\sigma_{xz}^{(i)}$ is interpolated using quadratic functions to obtain a parabolic distribution as follows:

$$\sigma_{xz}^{(i)}(\bar{z}) = \sum_{j=1}^{3} \phi_j(\bar{z}) \sigma_j^{(i)} \quad (9)$$

where $\bar{z}$ = a nondimensional local coordinate with origin at the bottom surface of the ith layer; $\sigma_1^{(i)}$, $\sigma_2^{(i)}$, and $\sigma_3^{(i)}$ = the shear stresses at the bottom, middle, and top of the ith layer; and ϕ_j = the second-order Lagrange polynomials, respectively. As mentioned in Chaudhuri and Seide (1987) and Reddy et al. (1989), the required number of equations are derived from the following conditions: Equating the average shear stress with the constant shear stress from constitutive equation on each layer yields

$$\sigma_{xz}^{(i)} = \frac{1}{6} [\sigma_1^{(i)} + 4\sigma_2^{(i)} + \sigma_3^{(i)}] \quad (10)$$

The shear-free condition on the top and bottom surfaces of the beam is

$$\sigma_1^{(i)} = \sigma_3^{(N)} = 0 \quad (11)$$

At the interfaces the continuity of shear stress satisfies the condition

$$\sigma_3^{(i)} - \sigma_1^{(i+1)} = 0 \quad (12)$$

Finally, the slope discontinuity of the shear stress at an interface becomes

$$\sigma_{xz,z}^{(i+1)} - \sigma_{xz,z}^{(i)} = \frac{1}{h^i} [-\sigma_1^{(i)} + 4\sigma_2^{(i)} - 3\sigma_3^{(i)}] + \frac{1}{h^{i+1}} [-3\sigma_1^{(i+1)} + 4\sigma_2^{(i+1)} - \sigma_3^{(i+1)}] \quad (13)$$

To satisfy stress equilibrium, the variation of the shear stress can be equated to the variation of the normal stress as

$$\sigma_{xz,z} = -\sigma_{xx,x} \quad (14)$$

By arranging (10)–(14) with respect to unknown $\sigma_j^{(i)}$, $3N$ simultaneous equations can be obtained, whose solution gives the parabolic distribution of shear stress. For convenience, the stresses calculated at the Gauss points are extrapolated to the nodes using linear interpolation functions, as suggested by Cook et al. (1989).

NUMERICAL EXAMPLES

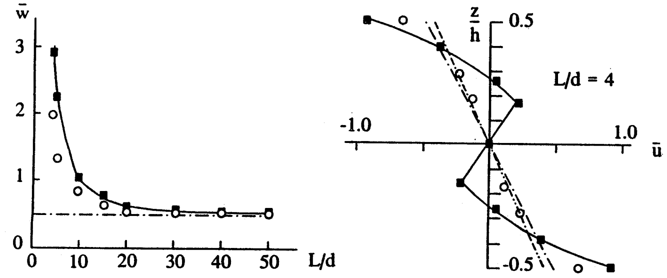
To illustrate the capability of the model and to verify its accuracy, two examples are solved and the results are compared with available solutions.

Elasticity-BLCS Comparisons

An elasticity solution of a cylindrical plate bending problem was presented by Pagano (1969). The BLCS predictions are compared with Pagano's elasticity and classical-plate-theory (CPT) solutions and with solutions by Pryor and Barker (1971) and model 4 of Kant and Manjunath (1989). Among the three laminate layouts ([0], [90/0], and [0/90/0]) considered by Pagano (1989), the solution for the [0/90/0] laminate with L/d = 4 is presented here. The unit-width beam with thickness h is subjected to a sinusoidal load of the form $\sin(\pi x/L)$. In the current study, four eight-layer elements are used to model the span of the beam. The nondimensional plots in Figs. 1 and 2 show that the results of the BLCS analysis agree well with the elasticity solution, while the other studies shown could not precisely represent the actual behavior.

Experimental-BLCS Stress Comparisons

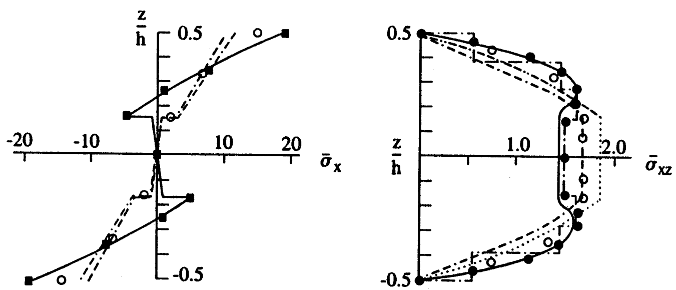
Using photoelasticity, Kemmochi and Uemura (1980) measured the stress distribution in four three-layered beams, which were 22 cm long and tested in bending under two symmetric point loads placed 11 cm apart. Three material types (A, B, and C) were used for the laminates, and the corresponding properties in megapascals (kips per square inch) are: E_A = 2,440 (354), G_A = 875 (127); E_B = 521 (75.5), G_B = 181 (26.2); and E_C = 4.34 (0.629), G_C = 0.53 (0.222). The material and thickness combinations of the core and face layers for the four beams are given in Table 1. Kemmochi and Uemura predicted the response of the test beams by a multilayer built-up theory. In BLCS, 16 eight-layer elements are used to compute the stresses



(a) transverse displacement at midspan (b) longitudinal displacement at support

—·—·· Pagano (CPT)
—— Pagano (Elasticity)
------ Pryor & Barker
○ Kant & Manjunath (model 4)
■ BLCS

FIG. 1. Displacements Comparison



(a) normal stress at midspan (b) shear stress at support

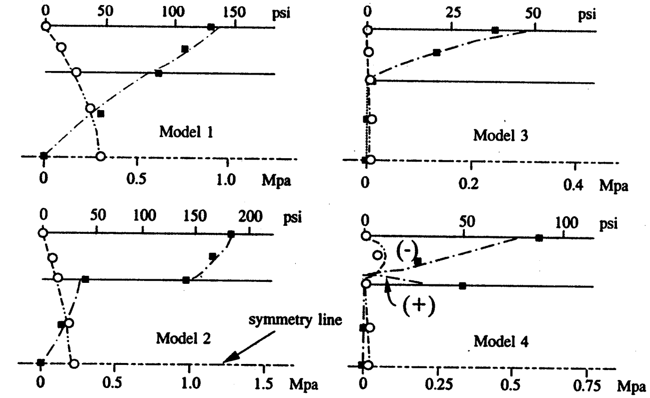
------ Pagano (CPT)
—— Pagano (Elasticity)
---·--- Pryor & Barker
○ Kant & Manjunath (model 4)
■ BLCS
—·—·— BLCS(constitutive equation)
● BLCS(equilibrium)

FIG. 2. Stresses Comparison

TABLE 1. Test Model Specification

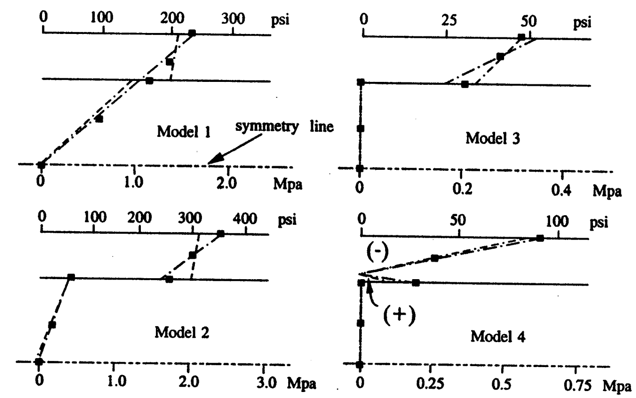
Model (1)	Material face/core (2)	Thickness, [mm (in.)]		Width, [mm (in.)] (5)	Load (*W*), [N (lb)] (6)
		Face (3)	Core (4)		
1	A/A	4.48 (0.176)	30.0 (1.18)	6.56 (0.258)	98 (22.03)
2	A/B	4.76 (0.187)	30.0 (1.18)	6.57 (0.259)[a]	98 (22.03)
3	B/C	4.51 (0.178)	30.0 (1.18)	6.32 (0.249)	9.8 (2.203)
4	A/C	4.72 (0.186)	30.0 (1.18)	6.57 (0.259)[a]	9.8 (2.203)

[a]In the experiment, the face width was a little larger than that of the core. The face width is used for core and face in BLCS.



Normal Shear
—·— ---·-- Kemmochi & Uemura(experiment)
■ ○ BLCS

FIG. 3. Stress Distribution at Section *s-s*



—·—·· Kemmochi & Uemura(experiment)
-·-·-- Kemmochi & Uemura(multi-layer built-up theory)
■ BLCS

FIG. 4. Normal Stress Distribution at Section *m-m*

at sections *s-s* (constant shear), located 3 cm from the support, and *m-m* (maximum moment), located at midspan. In Fig. 3, the BLCS and experimental normal- and shear-stress distributions are compared at section *s-s*, and in Fig. 4, the BLCS and multilayer built-up–theory normal-stress predictions are compared to experimental measurements at section *m-m*. The BLCS predictions agree closely with the experimental results. The BLCS accuracy for predicting the stress is particularly significant in model 4, which has a very low core stiffness compared to face stiffness.

SUMMARY AND CONCLUSION

A three-node laminated beam finite element with layerwise constant shear is formulated under the assumption of transverse incompressibility and layerwise linear distribution of in-plane displacement. Therefore, a cross section of the beam does not necessarily remain plane through the thickness but plane on each layer. The constitutive equation for a lamina is based on plane stress assumption, through both the thickness and width of the beam. The laminate constitutive equations are calculated using the constitutive equations and linear interpolation functions of in-plane displacements of the laminae. Using conditions of compatibility and equilibrium, the layer-wise constant shear distribution is modified into a parabolic distribution. Numerical and experimental solutions available in the literature are used to evaluate the accuracy of the BLCS element. Compared with the results of other studies, the prediction of displacements and stresses of the BLCS element are quite accurate.

APPENDIX. REFERENCES

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