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EULERIAN FINITE ELEMENT FORMULATION OF THE FLUID MECHANICS IN THE PULTRUSION PROCESS

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Abstract

A finite element model is developed to study the fluid mechanics of the pultrusion processing of fiber reinforced polymer composites. The problem is modelled as viscous creeping flow into an anisotropic porous media. The governing equations are recast in Eulerian coordinates to analyse the control volume defined by the interior of the die. The equations describing incompressible, creeping flow in a porous media are integrated over an averaging volume to produce Darcy's law. The movement of the fibers at constant speed through the control volume is properly accounted for, by transforming the Darcy's equation from Lagrangian coordinate to Eulerian cordinate. The finite element model is developed using variational approach and a linear quadrilateral element is used to produce particular solutions.

Nomenclature

- A Area of a layer of unit thickness
- F body forces
- k permeability of the medium
- k_x permeability of the medium in x- direction
- k_y permeability of the medium in y- direction
- N Interpolation functions
- p fluid pressure
- Q Volume flow
- q seepage velocity
- $\mathbf{v}$ velocity vector
- v_L velocity of the liquid
- v_s velocity of the solid
- v_o Velocity of approach
- v_p pulling velocity of the fibers
- α void distribution function
- μ Viscosity of the medium
- ρ Density of the medium
- ρ_L Density of the liquid
- ρ_s Density of the solid
- τ stresses in the fluid
- Δp pressure differential between inlet and outlet

1. Introduction

Pultrusion is a continuous manufacturing process for fiber reinforced polymer composites. The pultrusion process has gained acceptance in many new applications such as structural supports, window trim and automotive drive shafts. Many authors have studied the flow through porous media and phase change problems separately, but little attention has been given for modelling pultrusion process[1]. For a complete understanding of the problem, we need to solve a model which includes: (a) Fluid flow in a porous media (fiber glass filled die) (b) Heat transfer analysis with phase change and (c) Cure kinetics with heat of reaction and material parameters coupled with the heat transfer and fluid mechanics problem.

The flow of resin in the die is highly viscous mainly close to the transition zone. Darcy's law is considered valid for creeping flow where the Reynold's number defined for a porous medium[3] is less than 1.0. In this paper, the flow of resin and fibers in a die is modelled as a flow through an anisotropic porous medium using finite element method. Darcy's law in three dimensions is assumed to be valid inside the die.

2. Previous Work

Until recently only very few have analyzed pultrusion processing. The work was mostly experimental and little modelling was performed. Batch and Macosko[1] has given the details of previous study in pultrusion and they have also presented a model with certain limitations. A mechanistic kinetic model was described but it was not able to predict changes in curing rate after a change in resin formulation. The heat transfer model was developed using this kinetic. A pulling force model was suggested mainly for analysing the effect of friction between solid and wall. Whitaker[10] and Slattery[8] has presented the theory of creep flow in a porous media and the need for volume averaging. Scheidegger[5] was one of the earliest to present Darcy's law for the flow of homogeneous fluids in anisotropic porous media in a differential form.

Shamsunder and Sparrow[6] has given the details of previous work dealing with phase change heat conduction problems. The studies have mainly concentrated in predicting the location of the solid- liquid interface. The literature can be divided into two groups based on the choice of variables used. In the first group the temperature is the dependent variable and energy conservation equations are written separately in the solid region and in the liquid region. In the second group an enthalpy formulation is used. Shamsunder and Sparrow[6] have presented an enthalpy model for multidimensional phase change conduction problems. Storti,Crivelli and Idelsohn[7] have presented a numerical solution of phase change problems in two dimensions using a finite element technique. Although many authors have worked on flow through porous media and phase change problems separately, no literature is available combining both these together in Eulerian formulation to model a pultrusion process.

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3. Governing Equations

In this section we develop the governing equations for the flow of resin through a moving bed of fibers. The equations are cast in Eulerian coordinates with the domain being the interior of the die. Some of the concepts presented next are based on the work of Scheidegger[5], Whitaker[10], Slattery[8,9] and Greenkorn[3]. Darcy's law in differential form can be written as

$$\frac{Q}{A} = q = -\frac{k}{\mu}\frac{dp}{dx} \qquad (1)$$

where q is the seepage velocity which is equivalent to the velocity of approach v_o in soil mechanics problems, k is the intrinsic permeability of the medium, μ is the viscosity of the fluid and p is the fluid pressure. In three dimensions Darcy's law becomes[3]

$$q = -\frac{[k]}{\mu}\nabla p \qquad (2)$$

For the creep flow of a Newtonian fluid with constant viscosity (μ) in a rigid porous medium, the continuity equation of the fluid is

$$\frac{\partial \rho}{\partial t} + \rho(\nabla . \mathbf{v}) = 0 \qquad (3)$$

The equation of motion for the fluid is

$$\frac{\partial \rho \mathbf{v}}{\partial t} + (\nabla . \rho \mathbf{v}\mathbf{v}) - (\nabla . [\tau]) - \rho \mathbf{F} = 0 \qquad (4)$$

where v is the velocity vector, $[\tau]$ are the stresses in the fluid and **F** the body forces.
To solve these two equations, it is required to know which part of the domain is liquid, which can be established by a void distribution function.

$\alpha(\delta) = 1$ if δ is in the fluid
$\alpha(\delta) = 0$ if δ is in the solid

If the function $\alpha(\delta)$ were known it would be possible, in principle to solve the governing equations and thus determine completely the pressure and velocity fields. However $\alpha(\delta)$ is never known and we cannot solve equations (3) and (4). The method used to overcome this is to volume average the equations of change[10]. The volume average of a point quantity $[\psi]$ associated with the fluid is

$$\langle\psi\rangle = \frac{1}{V}\int_{V_L}\psi\, dv \qquad (5)$$

Whitaker[10] requires that

$$\langle\langle\psi\rangle\rangle = \langle\psi\rangle \qquad (6)$$

$$\text{where } \langle\langle\psi\rangle\rangle = \frac{1}{V}\int_{V}\langle\psi\rangle\, dV \qquad (7)$$

From a series expansion

$$\langle\langle\psi\rangle\rangle = \langle\psi\rangle + O[\langle\psi\rangle(\frac{l}{L})^2] \qquad (8)$$

using general transport theorem volume average of a gradient can be written as[9]

$$\langle\nabla\psi\rangle = (\nabla\langle\psi\rangle) + \frac{1}{V}\int_{A_i}(\psi n)\, dA \qquad (9)$$

where n is the outward directed normal for the total volume V or the liquid volume V_L and A_i is the area of the solid-fluid interface. The divergence is described by

$$\langle(\nabla . \psi)\rangle = (\nabla . \langle\psi\rangle) + \frac{1}{V}\int_{A_i}(\psi . n)\, dA \qquad (10)$$

and the mass balance for the fluid inside the closed surface S is

$$\int_{V_L}[\frac{\partial\rho}{\partial t} + (\nabla . \rho\mathbf{v})]\, dv = 0 \qquad (11)$$

Using Eq. (9) and (10) we obtain

$$\frac{\partial}{\partial t}\langle\rho\rangle + (\nabla . \langle\rho\mathbf{v}\rangle) = 0 \qquad (12)$$

The term $\langle\rho v\rangle$ represents the rate of mass flow per unit area through the porous media. The rate of flow per unit area $\langle v\rangle$ which is the same as the seepage velocity q, can be computed as

$$\langle\mathbf{v}\rangle = \frac{1}{V}\int_{V_L}\mathbf{v}_L dV_L + \frac{1}{V}\int_{V_S}\mathbf{v}_s dV_S \qquad (13)$$

where v_L is the velocity of the liquid and v_s is the velocity of solid, which is zero with respect to the material system moving with the fibers, therefore

$$\langle\mathbf{v}\rangle = \frac{1}{V}\int_{V_L}\mathbf{v}_L\phi\, dV \quad , \quad dV_L = \phi\, dV \qquad (14)$$

$$\mathbf{q} = \langle\mathbf{v}\rangle = \phi[\frac{1}{V}\int_{V}\mathbf{v}_L dV] \qquad (15)$$

The rate of mass flow per unit area is defined as

$$m = \rho_L q \qquad (16)$$

Replacing q from Eq. (15), Eq.(16) can be written as

$$m = \rho_L\phi\frac{1}{V}\int_{V}\mathbf{v}_L dV \qquad (17)$$

The continuity Eq.(12) contains the term $\langle\rho v\rangle$, which is the rate of mass flow per unit area

$$\langle\rho\mathbf{v}\rangle = m = \rho_L\mathbf{q} \qquad (18)$$

Therefore a fundamental question placed by Slattery[9] is resolved,

$$\langle\rho v\rangle = \rho_L\langle v\rangle \quad m = \rho_L q \tag{19}$$

If the fibers are moving with a constant velocty V_p then Eq.(2) can be recast in Eulerian coordinates and becomes

$$\langle v\rangle - V_p = -\frac{[k]}{\mu}\nabla p \tag{20}$$

$$\langle v\rangle = -\frac{[k]}{\mu}\nabla p + V_p \tag{21}$$

The continuity Eq.(12) becomes

$$\nabla.\langle\rho v\rangle = 0 \quad when \frac{\partial}{\partial t}\langle\rho\rangle = 0 \tag{22}$$

substituting v from Eq. (21) and using Eq.(18)

$$\begin{aligned}&\nabla.[\rho_L(-\frac{[k]}{\mu}\nabla p + V_p)] = 0\\ &-\nabla.(\rho_L\frac{[k]}{\mu}\nabla p) + \nabla.(\rho_L V_p) = 0\end{aligned} \tag{23}$$

using the identity

$$\begin{aligned}&\nabla.(\rho_L V_p) = (\nabla\rho_L).V_p + \rho_L\nabla.V_p\\ &= V_{p_x}\frac{\partial\rho}{\partial x} + V_{p_y}\frac{\partial\rho}{\partial y} + \rho\frac{\partial V_{p_x}}{\partial x} + \rho\frac{\partial V_{p_y}}{\partial y}\end{aligned} \tag{24}$$

equation (23) can be rewritten into the final governing equation,

$$-\nabla.(\frac{\rho k}{\mu}\nabla p) + V_{p_x}\frac{\partial\rho}{\partial x} + V_{p_y}\frac{\partial\rho}{\partial y} + \rho\frac{\partial V_{p_x}}{\partial x} + \rho\frac{\partial V_{p_y}}{\partial y} = 0 \tag{25}$$

4. Finite Element Formulation

Using a variational formulation, we can write the following functional[4]

$$\Psi = ½ B(p,p) - L(p)$$

$$\begin{aligned}\psi = &\frac{1}{2}\int_\Omega \frac{\rho k_x}{\mu}(\frac{\partial p}{\partial x})^2 dv + \frac{1}{2}\int_\Omega \frac{\rho k_y}{\mu}(\frac{\partial p}{\partial y})^2 dv\\ &+\int_\Omega pV_{p_x}\frac{\partial\rho}{\partial x}dV + \int_\Omega pV_{p_y}\frac{\partial\rho}{\partial y}dV\\ &+\int_\Omega p\rho\frac{\partial V_{p_x}}{\partial x}dV + \int_\Omega p\rho\frac{\partial V_{p_y}}{\partial y}dV - \int_L \bar{q}p ds\end{aligned} \tag{26}$$

where

$$\bar{q} = \frac{\rho k_x}{\mu}\frac{\partial p}{\partial x}n_x + \frac{\rho k_y}{\mu}\frac{\partial p}{\partial y}n_y \tag{27}$$

are the boundary terms. Equation (26) can be written in matrix form as

$$\begin{aligned}\psi = &\frac{1}{2}\int_\Omega\begin{bmatrix}\frac{\partial p}{\partial x} & \frac{\partial p}{\partial y}\end{bmatrix}\begin{bmatrix}\frac{\rho k_x}{\mu} & 0\\ 0 & \frac{\rho k_y}{\mu}\end{bmatrix}\begin{Bmatrix}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial y}\end{Bmatrix}dV\\ &+\int_\Omega pV_{p_x}\frac{\partial\rho}{\partial x}dV + \int_\Omega pV_{p_y}\frac{\partial\rho}{\partial y}dV\\ &+\int_\Omega p\rho\frac{\partial V_{p_x}}{\partial y}dV + \int_\Omega p\rho\frac{\partial V_{p_y}}{\partial y}dV - \int_L \bar{q}p ds\end{aligned} \tag{28}$$

Using a quadrilateral linear element with the pressure p inside the element approximated in terms of the pressure at the nodes {p}

$$p = [N]\{p\} \tag{29}$$

The velocity vector

$$\begin{Bmatrix}v_x\\ v_y\end{Bmatrix} = -\frac{1}{\mu}\begin{bmatrix}k_x & 0\\ 0 & k_y\end{bmatrix}\begin{Bmatrix}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial y}\end{Bmatrix} \tag{30}$$

can be written in compact form as

$$\{h\} = [R]\{g\} \tag{31}$$

where the pressure gradient

$$\begin{Bmatrix}\frac{\partial p}{\partial x}\\ \frac{\partial p}{\partial y}\end{Bmatrix} = \begin{bmatrix}\frac{\partial N_1}{\partial x} & \frac{\partial N_2}{\partial x} & \frac{\partial N_3}{\partial x} & \frac{\partial N_4}{\partial x}\\ \frac{\partial N_1}{\partial y} & \frac{\partial N_2}{\partial y} & \frac{\partial N_3}{\partial y} & \frac{\partial N_4}{\partial y}\end{bmatrix}\begin{Bmatrix}p_1\\ p_2\\ p_3\\ p_4\end{Bmatrix} \tag{32}$$

which can be written as

$$\{g\} = [B]\{p\} \tag{33}$$

where p is the pressure interpolated in terms of nodal values, {p} are the pressure at the nodes, and [N] are the Interpolation functions. Substituting Eq.(29), (31) and (33) into equation (28)

$$\begin{aligned}\psi = &\frac{1}{2}\int_\Omega\{p\}^T[B]^T[\rho R][B]\{p\}dv + \int_\Omega\{p\}^T[N]^T V_{p_x}\frac{\partial\rho}{\partial x}dV\\ &+\int_\Omega\{p\}^T[N]^T V_{p_y}\frac{\partial\rho}{\partial y}dV + \int_\Omega\{p\}^T[N]^T\rho\frac{\partial V_{p_x}}{\partial x}dV\\ &+\int_\Omega\{p\}^T[N]^T\rho\frac{\partial V_{p_y}}{\partial y}dV + \int_L\bar{q}p ds\end{aligned} \tag{34}$$

Minimising the functional ψ we obtain the finite element formulation

$$\begin{aligned}\int_\Omega[B]^T[\frac{\rho k}{\mu}][B]\{p\}dv = &-\int_\Omega[N]^T\{V_p\}^T[B]\{\rho\}dV\\ &-\int_\Omega[N]^T\rho\ tr([B][V_p])dV + \int_L\bar{q}ds\end{aligned} \tag{35}$$

[Vp] are the two components of the pulling velocity at the nodes and {Vp} are the two components of the pulling velocity in the domain computed as $\{Vp\}^T = [N][Vp].\{\rho\}$ are the nodal densities and ρ is the density inside the element computed as $\rho = [N]\{\rho\}$. The density ρ, viscosity μ and permeability tensor [k] at any point for an element is interpolated in terms of the nodal values using the interpolation functions. Vp along the fiber direction is constant and Vp_x and Vp_y are calculated from the geometry of the die. External forces such as sources and sinks can also be added to the formulation.

5. Results

A computer program has been developed for determining the pressure at the nodes, and velocity at the Gauss points of the elements. The program allows for variable density ρ, viscosity μ and permeability tensor k. Initially the program was checked for $V_p = 0.0$ using examples in standard text books. Excellent correlation with results in the open literature have been obtained. Next V_p was introduced and the variation of density and permeability was also considered. The results agrees very well with the physical nature of the problem. The results have been plotted using nondimensional quantities.

Example 1: Ground water flow or seepage[4] (Ex.4.3) is modelled using 8 elements and 15 nodes. Anisotropic permeability ($k_x = 2\ k_y = 40$ m/day) is used. There are two pumps (pump 1: 1200 m^3/day.m^3, pump 2: 2400 m^3/day.m^3)and a line source as shown in Fig. 1. The computed pressure contour plots are shown in Fig.2.

To demonstrate the capabilities of the program we present the following examples for tapered and non-tapered pultrusion die with the properties as shown in Table 1. Non-dimensionalised length, pressure and velocity are used for plotting the results. The normalization is done as follows:

Normalised length, $x' = x / L$

Normalised pressure, $P' = P / (D_3\ \rho_3)$

Normalised velocity, $v' = v / (D_3 / \rho_3)$

Example 2: In this example we analyse the rectangular die to illustrate the effects of Pulling velocity, variable density and pressure boundary condition.

We show the pressure distribution along the die in Fig.5 and the velocity distribution in Fig.6. For cases 1,2,8 and 9 the density ρ is constant and therefore the velocity is constant. Case 3 is a classical flow problem through fixed, constant permeability porous media. Hence the pressure is linear and the velocity is constant. In case 4 there is no pressure differential Δp between inlet and outlet and there is no pulling velocity Vp, therefore pressure is constant throughout and velocity is zero. We show the effect of variable density ρ in cases 5,6 and 7. Case 5 illustrates the effect of Vp for variable density. The effect of Vp is evident in case 7 for similar conditions to case 6 with the velocity increasing considerably due to the combined effects of Vp and Δp.

Example 3: To illustrate the effect of geometry of the die, pulling velocity, density, permeability and pressure boundary condition we analyze a tapered die.

In Fig.7 and Fig.9 we show the pressure distribution and in Fig.8 and Fig.10 we show the velocity distribution along the die. For comparison with the rectangular die we present Fig. 7 and 8. Most cases have constant [k] which is unrealistic for pultrusion but useful for comparison. Except for case 4, all the curves are different from Fig.5 and Fig.6. The effect of Vp for a tapered die is evident by comparing cases 2 and 3 in Fig.7 and 8 to the same cases in Fig.5 and 6 for a tapered die. It can be seen from Fig.7 that the pressure variation is nonlinear in most cases and from Fig.8 that the velocity increases to account for the taper.

Fig.9 and 10 represent realistic cases of pultrusion with variable density ρ and permeability k. The effect of variable permeability can be seen by comparing cases 1 and 8. The flow is due to the pulling velocity Vp only. In case 1, constant [k] is equivalent to a larger number of fibers at the entrance moving with Vp, so velocity is higher compared to case 8 that approximates the real case of constant number of fibers. It can be also seen that in rectangular die, cases 1 and 8 produce a linear pressure variation and a constant velocity. Therefore the pressure and velocity for cases 1 and 8 are same. The effect of Vp can be seen in case 12 for similar conditions to case 11, with the velocity increasing considerably. In case 10 there is no pressure differential Δp and the flow is due to pulling velocity Vp only. The effect of Vp with varying ρ and [k] is illustrated in this case. It can be concluded that the geometry has significant influence on pressure and velocity.

6. Future Study

The fluid density and viscosity are temperature dependent and the permeability depends on porosity. Therefore the heat transfer analysis also has to be incorporated to get the correct behavior. An enthalpy model will be used to study the phase change problem. At present, the no slip boundary condition at the wall cannot be enforced and the friction is neglected. As work progresses these will be also taken into consideration.

7. References

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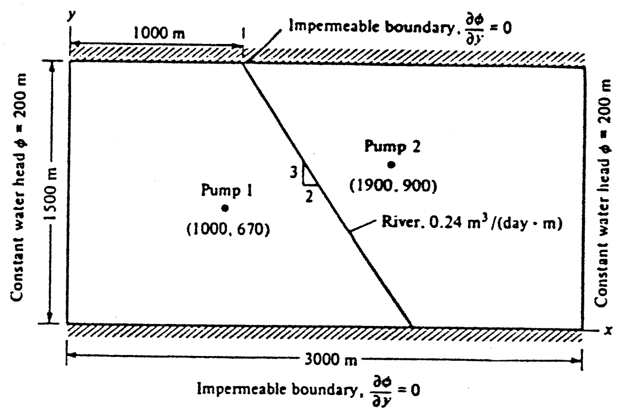

Fig. 1. Ground water flow[4] (Ex. 4.3)

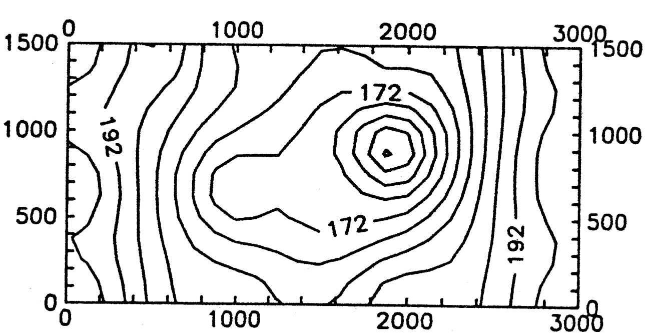

Fig. 2. Pressure contours for Ex.1

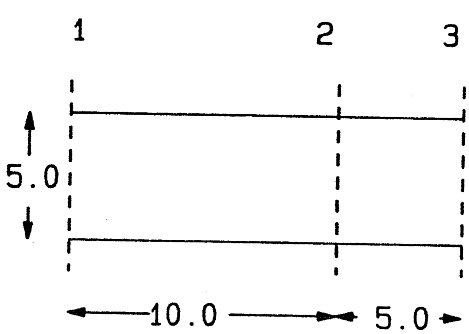

Fig. 3. Rectangular Die

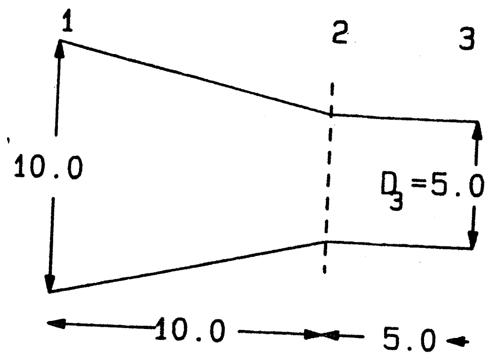

Fig. 4. Tapered Die

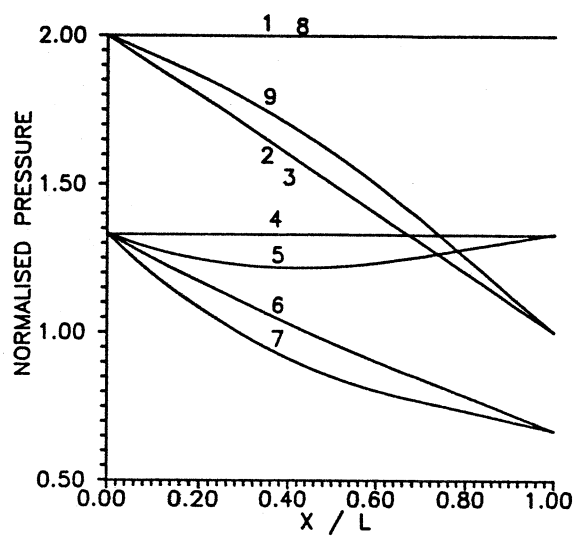

Fig.5. Pressure distribution on rectangular die

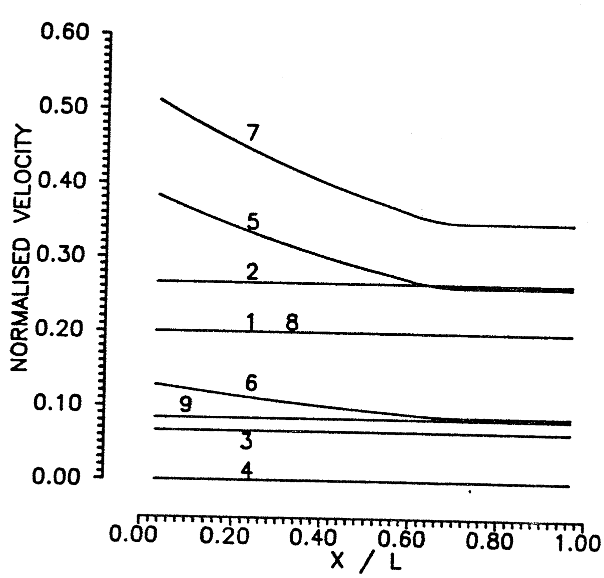

Fig.6. Velocity distribution on rectangular die

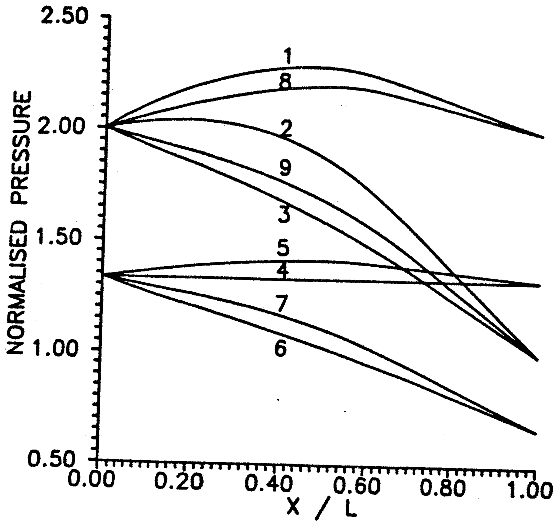

Fig.7. Pressure distribution for tapered die, comparison with rectangular die.

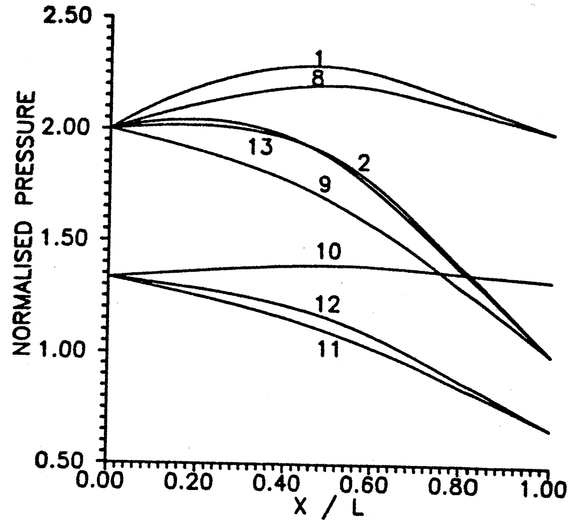

Fig.9. Pressure distribution for tapered die, effect of variable permeability.

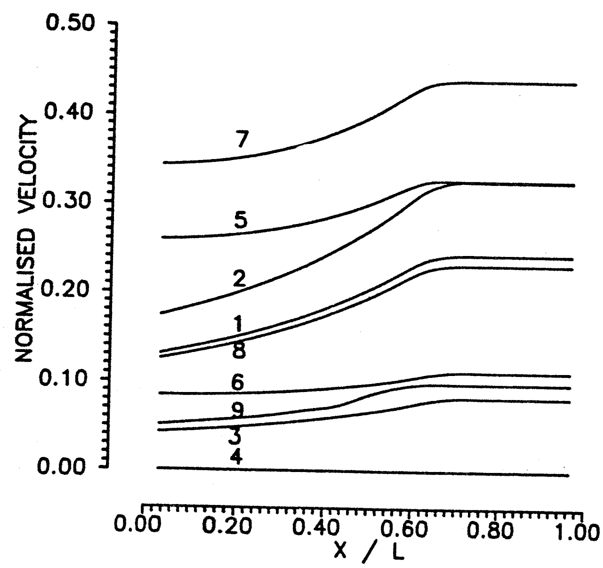

Fig.8. Velocity distribution for tapered die, comparison with rectangular die.

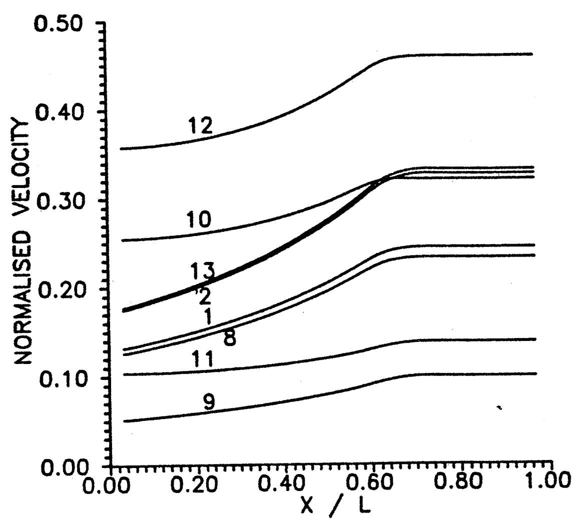

Fig.10. Velocity distribution for tapered die, effect of variable permeability.

<u>Table</u> 1: **parameters used on different cases**

Case #	Vp	ΔP 1 - 3	ρ 1- 2- 3	[k] 1- 2- 3
1	1.0	0.0	1.0- 1.0- 1.0	1.0- 1.0- 1.0
2	1.0	5.0	1.0- 1.0- 1.0	1.0- 1.0- 1.0
3	0.0	5.0	1.0- 1.0- 1.0	1.0- 1.0- 1.0
4	0.0	0.0	1.0- 1.5- 1.5	1.0- 1.0- 1.0
5	1.0	0.0	1.0- 1.5- 1.5	1.0- 1.0- 1.0
6	0.0	5.0	1.0- 1.5- 1.5	1.0- 1.0- 1.0
7	1.0	5.0	1.0- 1.5- 1.5	1.0- 1.0- 1.0
8	1.0	0.0	1.0- 1.0- 1.0	2.0- 1.0- 1.0
9	0.0	5.0	1.0- 1.0- 1.0	2.0- 1.0- 1.0
10	1.0	0.0	1.0- 1.5- 1.5	2.0- 1.0- 1.0
11	0.0	5.0	1.0- 1.5- 1.5	2.0- 1.0- 1.0
12	1.0	5.0	1.0- 1.5- 1.5	2.0- 1.0- 1.0
13	1.0	5.0	1.0- 1.5- 1.5	2.0- 1.0- 1.0